ON TORSION ANOMALOUS INTERSECTIONS (WITH AN APPENDIX BY P. PHILIPPON)

SARA CHECCOLI, FRANCESCO VENEZIANO, AND EVELINA VIADA

Abstract. A deep conjecture on torsion anomalous varieties states that if V is a weak-transverse variety in an abelian variety, then the complement V^{ta} of all V-torsion anomalous varieties is open and dense in V. We prove some cases of this conjecture. First this holds for a weak-transverse translate in an abelian variety. With a totally effective method, we prove that the V-torsion anomalous varieties of relative codimension one are non-dense in any weak-transverse variety V embedded in a product of elliptic curves with CM. As an immediate consequence we prove the conjecture for V of codimension 2 in a product of CM elliptic curves. We also point out some implications on the effective Mordell-Lang Conjecture and how the method generalises to abelian varieties.

1. Introduction

In this article, by *variety* we mean an algebraic variety defined over the algebraic numbers. We denote by G a semi-abelian variety defined over a number field k and by k_{tor} the field of definition of the torsion points of G. Let V be a subvariety of G. The variety V is a *translate*, respectively a *torsion variety*, if it is a finite union of translates of proper algebraic subgroups by points, respectively by torsion points.

An irreducible variety V is *transverse*, respectively *weak-transverse*, if it is not contained in any translate, respectively in any torsion variety.

Of course a torsion variety is in particular a translate, and a transverse variety is weak-transverse. In addition transverse implies non-translate, and weak-transverse implies non-torsion.

Notice that we are considering only proper subvarieties.

It is a natural problem to investigate when a geometric assumption on V is equivalent to the non-density of some special subsets of V. Several classical statements, such as, for instance, the Manin-Mumford, the Mordell-Lang and the Bogomolov Conjectures—nowadays theorems, are all of this nature.

More recently, new questions of similar type have been raised. The Zilber-Pink Conjecture asserts that, for a transverse variety V, the intersection of V with the union of all algebraic subgroups of codimension at least $\dim V + 1$ translated by points in a subgroup of finite rank, is non-dense in V.

This conjecture has been tackled from several points of view, but it has only been answered partially. For instance it is known for curves in some semi-abelian varieties.

Key words and phrases. Subvarieties of products of elliptic curves, Finitness of torsion anomalous intersections, Diophantine approximation.

All authors are supported by the SNSF.

E. Bombieri, D. Masser and U. Zannier in [BMZ07] give a new approach for general dimensions. They introduce the notions of anomalous and torsion anomalous subvarieties. In their definitions they always avoid points. For us points can be torsion anomalous, but not anomalous. This gives a perfect match with the CIT conjecture, as clarified below.

An irreducible subvariety Y of V is a *V-torsion anomalous* variety if

- Y is an irreducible component of $V \cap (B+\zeta)$ with $B+\zeta$ an irreducible torsion variety;
- the dimension of Y is larger than expected, i.e.

$$\operatorname{codim} Y < \operatorname{codim} V + \operatorname{codim} B.$$

The variety $B+\zeta$ is *minimal* for Y if it satisfies the above conditions and has minimal dimension. The *relative codimension* of Y is the codimension of Y in its minimal $B+\zeta$.

We say that Y is a *maximal V-torsion anomalous* variety if it is V-torsion anomalous and it is not contained in any V-torsion anomalous variety of strictly larger dimension.

The complement in V of the union of all V-torsion anomalous varieties is denoted by V^{ta}. Clearly V^{ta} is obtained removing from V all maximal V-torsion anomalous varieties. Again, for Bombieri, Masser and Zannier V^{ta} is the complement of the union of all V-torsion anomalous varieties of positive dimension.

Furthermore, an irreducible variety Y of positive dimension is *V-anomalous* if it is a component of $V \cap (B+p)$ with $B+p$ an irreducible translate and in addition Y has dimension larger than expected. The complement in V of the union of all V-anomalous varieties is denoted by V^{oa}.

Clearly points should be excluded from the definition of V-anomalous varieties because otherwise all points would be anomalous, making the notion uninteresting. On the other hand we allow points to be torsion anomalous varieties: they are exactly the torsion anomalous varieties which are not anomalous.

It may be possible that only some components of $V \cap (B+\zeta)$ are anomalous, so each component has to be treated separately. This justifies the assumption of Y being irreducible.

In the Torsion Openness Conjecture [BMZ07], Bombieri, Masser and Zannier conjectured that the complement of the set of the torsion anomalous varieties of positive dimension is open. In addition, in the Torsion Finiteness Conjecture, they claim that there are only finitely many maximal torsion anomalous points. Here we state a slightly stronger conjecture, which includes both their conjectures. In addition, it specifies that V^{ta} is empty exactly when V is not weak-transverse. In other words we say that there are only finitely many maximal torsion anomalous varieties of any dimension.

Conjecture 1 (Bombieri-Masser-Zannier)**.** *Let V be a weak-transverse variety in a (semi-)abelian variety. Then V^{ta} is a dense open subset of V.*

For a hypersurface the conjecture is clearly true. Indeed the intersection of an irreducible torsion variety $B+\zeta$ with a hypersurface is either the variety $B+\zeta$ itself or it has the right dimension $\dim B - 1$. So the only V-torsion anomalous varieties are torsion varieties contained in V; but we know by the Manin-Mumford Conjecture that the maximal torsion varieties contained in V are finitely many.

Among other results, Bombieri, Masser and Zannier in [BMZ07], Theorem 1.7, prove the openness of V^{ta} for an irreducible variety V of codimension 2 in $\mathbb{G}_m^n$. Among the main ingredients in their proof is a result of Ax consisting in the analogue of Schanuel's Conjecture in fields of complex power series in several variables.

In this paper, we first prove Conjecture 1 for weak-transverse translates in a general abelian variety. Then we give a totally effective method which shows that in any weak-transverse variety V in a product of elliptic curves with CM, the V-torsion anomalous varieties of relative codimension one are non-dense. An immediate consequence is Conjecture 1 for weak-transverse varieties of codimension 2, proving the analogue of Theorem 1.7 of [BMZ07] in a product of CM elliptic curves. Our method differs from theirs; in particular it is completely effective, and we also avoid the use of Ax's theorem. An intrinsic consequence of our effective bounds on the height is the effective CIT (stated below) for weak-transverse varieties of codimension 2 in a product of elliptic curves with CM. Finally we point out some implications on the effective Mordell-Lang Conjecture. We also show how the method for elliptic curves generalises to abelian varieties.

The following first result, based only on some geometric considerations, is proved in Section 3.4.

Theorem 1.1. *Let $H + p$ be a weak-transverse translate in an abelian variety. Then the set of $(H + p)$-torsion anomalous varieties is empty.*

This clarifies the situation and brings some simplifications in the proof of our main result:

Theorem 1.2. *Let V be a weak-transverse variety in a product of elliptic curves with CM defined over a number field k. Then the maximal V-torsion anomalous varieties Y of relative codimension one are finitely many; in addition their degree and normalised height are effectively bounded as*

$$h(Y) \ll_\eta (h(V) + \deg V)^{\frac{N-1}{N-1-\dim V}+\eta}[k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}]^{\frac{\dim V}{N-1-\dim V}+\eta},$$
$$\deg Y \ll_\eta (h(V) + \deg V)^{\frac{N-2}{N-1-\dim V}+\eta}[k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}]^{\frac{\dim V-1}{N-1-\dim V}+\eta}.$$

These are only some of the bounds we obtain. The method is totally effective, and we give explicit dependence on V. The notations will be made precise in the next sections.

In addition we bound the degree of the torsion varieties $B + \zeta$ minimal for the maximal V-torsion anomalous that we consider; these bounds provide, in principle, an algorithm to find all such anomalous varieties. The effective statements require some further notation and they are given in Theorem 5.1 for maximal torsion anomalous varieties which are not translates, in Theorems 6.1 and 6.2 for maximal torsion anomalous points and in Theorems 7.3 and 7.5 for maximal torsion anomalous translates of positive dimension.

The proof of our main theorem has four main ingredients: the deep Zhang inequality, the strong explicit Arithmetic Bézout theorem by Philippon, a sharper variant of an effective result by Galateau on the Bogomolov Conjecture, and the relative Lehmer bound in CM abelian varieties by Carrizosa.

In the proofs we need to distinguish whether a V-torsion anomalous variety Y is or not itself a translate.

If Y is not a translate then the Zhang inequality, the explicit Arithmetic Bézout theorem, and a functorial version of the effective Bogomolov Conjecture are sufficient. This case is proved in a product of elliptic curves, as the Bogomolov type bounds do not require any assumption on complex multiplication.

If Y is a point we use the Zhang inequality, the explicit Arithmetic Bézout theorem and the relative Lehmer bound; here we need to assume that the elliptic curve has CM, as a sharp Lehmer type bound is known only under this hypothesis.

Finally, we reduce the case of translates of positive dimension to the case of points. To this aim we also use a result recalled in the appendix by Patrice Philippon, which relates the essential minimum of a translate to the height of the point of translation.

Theorem 1.2 can be easily generalised to abelian varieties, in the following sense.

Theorem 1.3. *Let V be a weak-transverse variety in an abelian variety A with CM. Let g be the maximal dimension of a simple factor of A. If $\operatorname{codim} V \geq g+1$, then the maximal V-torsion anomalous varieties of relative codimension 1 are finitely many and they have degree and normalised height effectively bounded.*

For clarity, we first present the proof of Theorem 1.2, where the technicalities are simpler to follow. We then explain how this generalises to abelian varieties in Section 8.

We notice that the most interesting case remains the case of elliptic curves, in the sense that there is the largest number of subgroups. A breaking-through result would be the proof of Theorem 1.2 for any relative codimension with an effective method. Such a general result would imply the effective Mordell-Lang Conjecture.

We now give several consequences and applications of our main theorem. By definition all V-torsion anomalous varieties satisfy a dimensional inequality. As a straightforward consequence, we see that in a variety V of codimension 2 the relative codimension of a V-torsion anomalous variety is either one or zero. Then, from Theorem 1.2, it follows immediately the following:

Theorem 1.4. *Let V be a weak-transverse variety of codimension 2 in a product of elliptic curves with CM. Then V^{ta} is a dense open subset of V.*

Proof. It is sufficient to consider maximal V-torsion anomalous varieties. Let Y be a maximal V-torsion anomalous component of $V \cap (B+\zeta)$.

Then, by definition of V-torsion anomalous variety,

$$\operatorname{codim} Y < \operatorname{codim} V + \operatorname{codim} B.$$

Equivalently

$$\dim B - \dim Y < \operatorname{codim} V = 2.$$

If Y has relative codimension zero, then $Y = B+\zeta$ and Y is a component of the closure of the torsion contained in V, which is a proper closed by the Manin-Mumford Conjecture. If Y has relative codimension one, we apply our main theorem. $\square$

Conjecture 1 is well known to be related to the following conjecture, which in turn is equivalent to the Zilber-Pink Conjecture. For a natural number r, define

$$S_r(V) = V \cap \bigcup_{\text{codim } H \geq r} H,$$

where H runs over all algebraic subgroups of codimension at least r.

Conjecture 2 (CIT, Conjecture on the Intersection with Torsion varieties)**.** *Let V be a weak-transverse variety in an abelian variety. Then $S_{\dim V+1}(V)$ is non-dense in V.*

By definition, for any torsion variety $B+\zeta$, the intersection $V^{ta} \cap (B+\zeta)$ has the right dimension. In particular, if a point of V lies in some algebraic subgroup of codimension $\geq \dim V + 1$, then that point is contained in a V-torsion anomalous variety (we would expect empty intersection), and so it does not belong to V^{ta}. Then, as a consequence of the effective version of Theorems 1.1 and 1.4, we obtain a completely effective version of the following theorem.

Theorem 1.5. *Let V be a weak-transverse variety of codimension 2 or a weak-transverse translate in a product of elliptic curves with CM. Then $S_{\dim V+1}(V)$ is non-dense in V and its closure is $V \setminus V^{ta}$.*

Proof. If $\operatorname{codim}(B+\zeta) \geq \dim V + 1$, then all components of $V \cap (B+\zeta)$ are torsion anomalous so they do not intersect V^{ta}. Therefore

$$V^{ta} \cap \bigcup_{\text{codim } H \geq \dim V+1} H = \varnothing$$

and $S_{\dim V+1}(V) \subseteq V \setminus V^{ta}$. By Theorems 1.1 and 1.4, V^{ta} is an open dense set in V. That the closure of $S_{\dim V+1}(V)$ is $V \setminus V^{ta}$ is proven exactly as in [BMZ07], page 25, for tori. Recall that for them points are not torsion anomalous varieties. □

As an immediate corollary we obtain:

Corollary 1.6. *Let C be a weak-transverse curve in E^3 where E is an elliptic curve with CM defined over a number field k. Then*

$$S_2(C) = C \setminus C^{ta}$$

is a finite set of cardinality and Néron-Tate height effectively bounded. In particular every non-torsion point $Y_0 \in S_2(C)$ satisfies

$$\hat{h}(Y_0) \ll_\eta (h(C) + \deg C)^{2+\eta}[k_{\text{tor}}(C) : k_{\text{tor}}]^{1+\eta},$$
$$[k(Y_0) : \mathbb{Q}] \ll_\eta (\deg C(h(C) + \deg C)[k_{\text{tor}}(C) : k_{\text{tor}}][k(C) : k])^{2+\eta}.$$

That $S_2(C)$ is a finite set is known for weak-transverse curves in tori ([Mau08], Theorem 1.2) and in any product of elliptic curves (see [Via08]). That the height is bounded is proved using the Vojta inequality in a non-effective way. Effective bounds for the height of $S_2(C)$ are given for weak-transverse curves in tori in [BHMZ10], using an effective Mordell-Lang Theorem, and in abelian varieties only for transverse curves in a product of elliptic curves in [Via03]. So this corollary is a first example of an effective bound for the height for a weak-transverse curve in abelian varieties.

In higher dimensions, the Bounded Height Conjecture proved by Habegger ([Hab09], theorem at page 407), together with the Effective Bogomolov Bound by Galateau

([Gal10], Theorem 1.1) and the non-density Theorem by Viada ([Via10], Theorem 1.6) imply the CIT for varieties with V^{ta} non empty embedded in certain abelian varieties which include all CM abelian varieties. The original formulation of the Bounded Height Theorem in [Hab08] did not contain an explicit height bound and the author did not discuss the effectivity of the result (however, in a forthcoming paper, Habegger provides an explicit version of the Bounded Height Theorem for tori). Even if the method is made effective, the assumption $V^{ta} \neq \varnothing$ is stronger than transversality. In this respect, ours is a new effective method in the context of the CIT for weak-transverse varieties.

Many are the contributions on the CIT of several authors. For a more extensive list, we refer to the references given in the papers mentioned above.

In the last part of the paper we generalise our method, obtaining an effective weak result for curves, related to the CIT. Theorem 9.1 gives a complete effective version of the following result.

Theorem 1.7. *Let C be a weak-transverse curve in E^N, where E is an elliptic curve with CM defined over a number field k. Then,*

$$C \cap \cup_{\operatorname{codim} H > \dim H} H$$

is a finite set of cardinality and Néron-Tate height effectively bounded. Here H ranges over all algebraic subgroups of codimension larger than their dimension. In particular every non-torsion point $Y_0 \in C \cap \cup_{\operatorname{codim} H > \dim H} H$ satisfies

$$\hat{h}(Y_0) \ll_\eta (h(C) + \deg C)^{\frac{N+1}{2}+\eta} [k_{\rm tor}(C) : k_{\rm tor}]^{\frac{N-1}{2}+\eta},$$

$$[k(Y_0) : \mathbb{Q}] \ll_\eta (\deg C[k(C) : k])^{\frac{N+1}{N-1}+\eta} ((h(C) + \deg C)[k_{\rm tor}(C) : k_{\rm tor}])^{\frac{N+1}{2}+\eta}.$$

In the following section we emphasise the implications of these theorems on the effective and quantitative Mordell-Lang Conjecture.

2. Applications to the Mordell-Lang Conjecture

2.1. **Applications to the effective Mordell-Lang Conjecture.** The CIT is well known to have implications on the Mordell-Lang Conjecture. The toric case of this conjecture has been extensively studied, also in its effective form, by many authors. A completely effective version in the toric case can be found in [BG06], Theorem 5.4.5, and generalisations in [BGEP09]. However an effective general result in abelian varieties is not known.

Since we prove an effective version of Theorem 1.2, we obtain the following effective cases of the Mordell-Lang Conjecture for curves in products of elliptic curves with CM, where effective means that we give a bound for the height of the set of points in C and in a group of finite rank Γ; the dependence on C and Γ is completely explicit. The assumption on the relative codimension gives a condition on the rank of Γ.

This method is completely different from the two classical effective methods known in abelian varieties. The method by Chabauty and Coleman, surveyed, for example, in [PM10], gives a Mordell-Lang statement for a curve in its Jacobian and Γ of rank less than the genus; it involves a Selmer group calculation and smart computations with the Jacobian of the curve, and it can be made effective in some particular cases (see [McC94]). The method by Manin and Demjanenko, described in Chapter 5.2 of [Ser89], gives an effective Mordell theorem for curves with many

independent morphisms to an abelian variety. As remarked by Serre, the method remains of difficult application.

Let E be an elliptic curve defined over the algebraic numbers.

We let $\hat{h}$ be the standard Néron-Tate height on E^N; if V is a subvariety of E^N, we shall denote by $h(V)$ the normalized height of V, as defined in [Phi91]. The height of a set is as usual the supremum of the heights of its points. If E is defined over a field k, we denote by k_{tor} the field of definition of all torsion points of E. All the constants in the following theorems become explicit if the constant for the Lehmer type bound of Carrizosa in [Car09] is made explicit.

Theorem 2.1. *Let C be a weak-transverse curve in E^N, with E a CM elliptic curve and $N > 2$. Let k be a field of definition for E. Let Γ be a subgroup of E^N such that the group generated by its coordinates is an* $\mathrm{End}(E)$*-module of rank one. Then, for any positive η, there exists a constant c_1, depending only on E^N and η, such that the set*

$$C \cap \Gamma$$

has Néron-Tate height bounded as

$$\hat{h}(C \cap \Gamma) \leq c_1 (h(C) + \deg C)^{\frac{N-1}{N-2}+\eta} [k_{\text{tor}}(C) : k_{\text{tor}}]^{\frac{1}{N-2}+\eta}.$$

Proof. Let g be a generator of $\overline{\Gamma}$, the group generated by all coordinates of any element in Γ. If a point $x = (x_1, \ldots, x_N)$ is in Γ then there exist $0 \neq a_i, b_i \in \mathrm{End}(E)$ and torsion points $\zeta_i \in E$ such that

$$a_i x_i = b_i g + \zeta_i, \qquad i = 1, \ldots, N.$$

If all $b_i = 0$ then x is a torsion point, thus it has height zero. We can suppose, for instance, that $b_1 \neq 0$. So g depends on x_1. Substituting this dependence in the last $N-1$ equations, we obtain a system of $N-1$ linearly independent equations in the variables $x_1, \ldots, x_N$ given by

$$b_1 a_j x_j = a_1 b_j x_1 + \zeta_j', \qquad j = 2, \ldots, N$$

for ζ_j' torsion points. These equations define a torsion variety H of codimension $N-1$ in E^N. Thus,

$$(C \cap \Gamma) \subseteq (C \cap \cup_{\dim H = 1} H) \cup (C \cap \mathrm{Tor}_{E^N}) = S_{N-1}(C)$$

for H ranging over all algebraic subgroup of dimension one. However, if $N > 2$, any point x in the intersection $C \cap H$ is a C-torsion anomalous point. In addition, as C is weak-transverse, each such C-torsion anomalous point is maximal. If x is not a torsion point then H is minimal for x, because H has dimension one. Thus the relative codimension of x in H is one. Applying Theorem 6.1 we deduce the bound. □

If $N = 2$, the intersection $C \cap H$ is not torsion anomalous, so we must follow another line. For this reason, we need the assumption of C being transverse.

Theorem 2.2. *Let C be a transverse curve in E^2 with E a CM elliptic curve defined over a number field k. Let Γ be a subgroup of E^2 such that the group of its coordinates is an* $\mathrm{End}(E)$*-module of rank one, generated by g. Then, for any positive η there exists a constant c_2 depending only on E^N and η, such that the set*

$$C \cap \Gamma$$

has Néron-Tate height bounded as

$$\hat{h}(C \cap \Gamma) \leq c_2 [k_{\text{tor}}(C \times g) : k_{\text{tor}}]^{1+\eta} (h(C) + (\hat{h}(g) + 1) \deg C)^{2+\eta}.$$

Proof. Let g be a generator of $\overline{\Gamma}$, the group generated by all coordinates of any element in Γ. Consider the curve $C' = C \times g$ in E^3. Since C is transverse then C' is weak-transverse. If a point (x_1, x_2) is in Γ then there exist $0 \neq a_i, b_i \in \mathrm{End}(E)$ and torsion points ζ_i such that

$$a_1 x_1 = b_1 g + \zeta_1, \quad a_2 x_2 = b_2 g + \zeta_2.$$

Thus the point (x_1, x_2, g) belongs to the intersection $C' \cap H$ with H the torsion variety of codimension 2 in E^3 defined by the equations

$$a_1 x_1 = b_1 x_3 + \zeta_1, \quad a_2 x_2 = b_2 x_3 + \zeta_2.$$

Thus $(x_1, x_2, g) \in (C' \cap \cup_{\dim H=1} H)$ for H ranging over all algebraic subgroups of dimension one in E^3. Therefore $C \cap \Gamma$ is embedded in $C' \cap \cup_{\dim H=1} H$ and $\hat{h}(C \cap \Gamma) \leq \hat{h}(C' \cap \cup_{\dim H=1} H)$. However any point x in the intersection $C' \cap H$ is a maximal C'-torsion anomalous point. If x is not a torsion point, then it has relative codimension one in H.

We apply Theorem 6.1 to $C' \subseteq E^3$, with $\deg C' = \deg C$ and

$$h(C') \leq 2(h(C) + \hat{h}(g) \deg C)$$

by Zhang's inequality, recalled in Section 4.3 below. This gives the bound for the height of a point in $C \cap \Gamma$. □

The assumptions on the curve are necessary in both theorems. Indeed for a weak-transverse curve C in E^2 the above theorem is not true. Consider the weak-transverse curve $C = E \times g$; for any positive integer m, let Γ_m be generated by the point $\gamma_m = (mg, g)$. For any m, the point γ_m belongs to C and, for m which goes to infinity, the height of γ_m tends to infinity. Thus there cannot be general bounds independent of Γ.

This does not happen in higher codimension. The analogue would be $C = E \times g \times g'$ where g and g' are linearly independent to ensure the weak-transversality. No point of C can be in Γ, as no point of the type (x, g, g') has coordinates in $\overline{\Gamma}$, which has rank one.

We also remark that for C weak-transverse, $C \times g$ is not necessarily weak-transverse: let $C = E \times g$, then $E \times g \times g$ is contained in the abelian subvariety $x_2 = x_3$.

With this method, the bound in the case $N = 2$ depends on g. It is possible to remove such a dependence. However one has to prove that for a transverse curve C, the set $C \cap \cup_{\dim H=1} H$ has bounded height. This is proved in [Via03] Theorem 1. The proof is effective, but not explicit in the dependence on C and on E. The dependence on C is given comparing the height function relative to a point with the height function relative to the intersections of $C \cap \Delta^-$ where Δ^- is the divisor defined by $x_1 = -x_2$. We use [Via03] to deduce another effective case of the Mordell-Lang Conjecture.

Theorem 2.3. *Let C be a transverse curve in E^N with E an elliptic curve defined over a number field. Let Γ be a subgroup of E^N such that the group of its coordinates is an* $\mathrm{End}(E)$*-module of rank at most $N-1$. Then the set*

$$C \cap \Gamma$$

is finite of Néron-Tate height bounded by a constant $c(C, E^N)$ *depending on* C *and* E^N *i.e.*

$$\hat{h}(C \cap \Gamma) \leq c(C, E^N).$$

Proof. The proof follows the same idea as the proof of Theorem 2.1. Let $g_1, \dots, g_{N-1}$ be generators of $\overline{\Gamma}$, the group generated by all the coordinates of any element in Γ. If a point $(x_1, \dots, x_N)$ is in Γ then there exist $a_i \in \mathrm{End}(E)$, an $N \times (N-1)$ matrix B with coefficients in $\mathrm{End}(E)$ and a torsion point $\zeta \in E^N$ such that

$$(a_1x_1, \dots, a_Nx_N)^t = B \cdot (g_1, \dots, g_{N-1})^t + \zeta.$$

Explicitating $(g_1, \dots, g_{N-1})$ in terms of $x_1, \dots, x_{N-1}$ from the first $N-1$ equations and substituting in the last equation, we obtain one equation in the variables $x_1, \dots, x_N$

$$a_1'x_1 + \cdots + a_N'x_N = \zeta',$$

where $\zeta' \in E$ is a torsion point and $a_i' \in \mathrm{End}(E)$. This equation defines a torsion variety H of codimension one in E^N. Thus,

$$(C \cap \Gamma) \subseteq (C \cap \cup_{\mathrm{codim}\, H \geq 1} H) = S_1(C)$$

for H ranging over all algebraic subgroups of codimension at least one. By [Via03] Theorem 1, this set has height bounded by an effective constant depending only on E^N and C. $\square$

At this point we want to extend our main theorem, at least in the case of curves, to see if with our method we could prove other cases of the Mordell-Lang Conjecture. Theorem 1.7 enables us to obtain a more general version of Theorem 2.1 for C weak-transverse and larger rank of $\overline{\Gamma}$. We also obtain a theorem similar to Theorem 2.3, with explicit dependence on C.

Theorem 2.4. *Let* C *be a weak-transverse curve in* E^N *with* E *an elliptic curve with CM. Let* k *be a field of definition for* E. *Let* Γ *be a subgroup of* E^N *such that the group of its coordinates is an* $\mathrm{End}(E)$*-module of rank* $t < N/2$. *Then, for any positive* η, *there exists a constant* c_3 *depending only on* E^N *and* η, *such that the set*

$$C \cap \Gamma$$

has Néron-Tate height bounded as

$$\hat{h}(C \cap \Gamma) \leq c_3(h(C) + \deg C)^{\frac{N-t}{N-2t}+\eta}[k_{\mathrm{tor}}(C) : k_{\mathrm{tor}}]^{\frac{t}{N-2t}+\eta}.$$

Proof. Let $g_1, \dots, g_t$ be generators of the free part of $\overline{\Gamma}$, the group generated by all coordinates of any element in Γ. If a point $(x_1, \dots, x_N)$ is in Γ then there exist $0 \neq a_i \in \mathrm{End}(E)$, an $N \times t$ matrix B with coefficients in $\mathrm{End}(E)$ and a torsion point $\zeta \in E^N$ such that

$$(a_1x_1, \dots, a_Nx_N)^t = B(g_1, \dots, g_t)^t + \zeta.$$

If the rank of B is zero, then x is a torsion point and so it has height zero.

If B has positive rank m, we can choose m equations of the system corresponding to m linearly independent rows of B. We use these equations to write the g_j in terms of the x_i and we substitute these expressions in the remaining equations.

We obtain a system of maximal rank with $N-m \geq N-t$ linearly independent equations in the variables $x_1, \dots, x_N$:

$$\begin{cases} a'_{11}x_1 + \cdots + a'_{1N}x_N = \zeta'_1 \\ \vdots \\ a'_{N-m,1}x_1 + \cdots + a'_{N-m,N}x_N = \zeta'_{N-m} \end{cases}$$

where $\zeta'_i \in E$ are torsion points and $a'_{ij} \in \mathrm{End}(E)$. These equations define a torsion variety H of codimension $N-m$ in E^N. Thus

$$(C \cap \Gamma) \subseteq S_{N-t}(C)$$

for H ranging over all algebraic subgroups of codimension at least $N-t$. However $\dim H = N - \operatorname{codim} H = m \leq t$. Thus, if $N > 2t$

$$S_{N-t}(C) \subseteq (C \cap \cup_{\operatorname{codim} H > \dim H} H).$$

Applying Corollary 9.2 with $r = N-t$ gives the wished bound for the height of $C \cap \Gamma$. $\square$

We notice that Theorems 2.1 and 2.4 are proved for weak-transverse curves, while previous effective results assumed transversality. If we assume the transversality of C we can relax the hypothesis on the rank of $\overline{\Gamma}$.

Theorem 2.5. *Let C be a transverse curve in E^N with E a CM elliptic curve defined over k. Let Γ be a subgroup of E^N such that the free part of the group of its coordinates is an* $\mathrm{End}(E)$*-module of rank $t \leq N-1$, generated by $g_1, \dots, g_t$. Then, for any positive η there exists a constant c_4 depending only on E^N and η, such that the set*

$$C \cap \Gamma$$

has Néron-Tate height bounded as

$$\hat{h}(C \cap \Gamma) \leq c_4 [k_{\mathrm{tor}}(C \times g) : k_{\mathrm{tor}}]^{\frac{t}{N-t}+\eta} (h(C) + (\hat{h}(g)+1) \deg C)^{\frac{N}{N-t}+\eta}$$

where $g = (g_1, \dots, g_t)$.

Proof. Consider the curve $C' = C \times g$ in E^{N+t}. Since C is transverse then C' is weak-transverse. If a point $(x_1, \dots, x_N)$ is in Γ, then there exist $0 \neq a_i \in \mathrm{End}(E)$, an $N \times t$ matrix B with coefficients in $\mathrm{End}(E)$ and a torsion point $\zeta \in E^N$ such that

$$(a_1 x_1, \dots, a_N x_N)^t = B(g_1, \dots, g_t)^t + \zeta.$$

Thus the point $(x_1, \dots x_N, g_1, \dots, g_t)$ belongs to the intersection $C' \cap H$ with H the torsion variety of codimension N and dimension t in E^{N+t} defined by the equations

$$(a_1 x_1, \dots, a_N x_N)^t = B(y_{N+1}, \cdots, y_{N+t})^t + \zeta.$$

Thus, $C \cap \Gamma$ is embedded in $C' \cap \cup_{\dim H = t} H$, and $\hat{h}(C \cap \Gamma) \leq \hat{h}(C' \cap \cup_{\dim H = t} H)$ for H ranging over all algebraic subgroups of dimension t. If $N > t$ then $\operatorname{codim} H > \dim H$.

The bound for the height is then given by Corollary 9.2 applied to $C' \subseteq E^{N+t}$, where $\deg C = \deg C'$, $h(C') \leq 2(h(C) + \hat{h}(g) \deg C)$ by Zhang's inequality, recalled in Section 4.3, below. $\square$

2.2. **Applications to the quantitative Mordell-Lang Conjecture.** In [Rém00], Theorem 1.2 Rémond gives a bound on the cardinality of the intersection $C \cap \Gamma$ for a transverse curve in E^N and Γ a $\mathbb{Z}$-module of rank r. He obtains the following bound

$$\#(C \cap \Gamma) \leq c(E^N, \mathcal{L})^{r+1} (\deg C)^{(r+1)N^{20}},$$

where $c(E^N, \mathcal{L})$ is a positive effective constant depending on E^N and on the choice of an invertible, symmetric and ample sheaf $\mathcal{L}$ on E^N.

The Manin-Mumford Conjecture is a special case of the Mordell-Lang conjecture. Explicit bounds on the number of torsion points in C are given, for instance, by David and Philippon in [DP07] and by Hrushovski in [Hru01]. David and Philippon in [DP07], Proposition 1.12 show that the number of torsion points on a non-torsion curve C is at most

$$\#(C \cap \mathrm{Tor}_{E^N}) \leq (10^{2N+53} \deg C)^{33},$$

where Tor_{E^N} is the set of all torsion points of E^N.

Our method enables us to give a sharp bound for the number of non torsion points in $C \cap \Gamma$ for C weak-transverse in E^N, which, together with the just mentioned bounds for the torsion, improves in some cases the bounds of Rémond. Notice that below we use the rank t of $\overline{\Gamma}$, the $\mathrm{End}(E)$-module of the coordinates of Γ. To compare with Rémond's result, we can use the trivial relation $r < 2Nt$ and $t < Nr$.

The following theorem is obtained combining the results from Section 2.1 with bounds from Theorems 6.2 and Corollary 9.2.

Theorem 2.6. *Let C be a curve in E^N, where E has CM and is defined over a number field k. Let Γ be a subgroup of E^N such that the group $\overline{\Gamma}$ of its coordinates is an $\mathrm{End}(E)$-module of rank t. Let S be the number of non-torsion points in the intersection $C \cap \Gamma$. Then, for every positive real η there exist constants d_1, d_2, d_3, d_4 depending only on E^N and η, such that:*

i. if C is weak-transverse in E^N, $N > 2$ and $t = 1$, we have

$$S \leq d_1((h(C) + \deg C)[k_{\mathrm{tor}}(C) : k_{\mathrm{tor}}])^{\frac{(N-1)(4N^2-N-4)}{2(N-2)^2}+\eta} (\deg C)^{\frac{2N^3-N^2+N-4}{2(N-2)}+\eta} \cdot [k(C) : k]^{\frac{N(N-1)(2N+1)}{2(N-2)}+\eta};$$

ii. if C is transverse in E^2 and $t = 1$, we have

$$S \leq d_2([k_{\mathrm{tor}}(C \times g) : k_{\mathrm{tor}}](h(C) + (\hat{h}(g) + 1) \deg C))^{29+\eta} (\deg C)^{22+\eta} \cdot [k(C \times g) : k]^{21+\eta}$$

where g is a generator of $\overline{\Gamma}$;

iii. if C is weak-transverse in E^N and $t < N/2$, we have

$$S \leq d_3((h(C) + \deg C)[k_{\mathrm{tor}}(C) : k_{\mathrm{tor}}])^{\frac{t(N-t)(4N^2-2Nt+N-2t-2)}{2(N-2t)(N-t-1)}+\eta} (\deg C)^{1+\frac{N(2N+1)(N-t)}{2(N-t-1)}+\eta} \cdot [k(C) : k]^{\frac{N(2N+1)(N-t)}{2(N-t-1)}+\eta};$$

iv. if C is transverse in E^N and $t \leq N-1$, we have

$$S \leq d_4([k_{\mathrm{tor}}(C \times g) : k_{\mathrm{tor}}](h(C) + (\hat{h}(g) + 1) \deg C))^{\frac{Nt(4N^2+2t^2+6Nt+N-t-2)}{2(N-t)(N-1)}+\eta} \cdot (\deg C)^{1+\frac{(N+t)N(2N+2t+1)}{2(N-1)}+\eta} [k(C \times g) : k]^{\frac{(N+t)N(2N+2t+1)}{2(N-1)}+\eta},$$

where $\overline{\Gamma}$ is generated by $g_1, \dots, g_t$ and $g = (g_1, \dots, g_t)$.

Proof. Cases (i) and (ii) follow from the proofs of Theorem 2.1 and Theorem 2.2, respectively, together with the bounds from Theorem 6.2. Cases (iii) and (iv) follow from the proofs of Theorem 2.4 and Theorem 2.5, respectively, together with the bounds from Corollary 9.2. □

3. Torsion anomalous varieties: preliminary results

In this section we denote by G an abelian variety or a torus and by Tor_G the set of all torsion points in G. In the next statements we consider subvarieties V of G.

3.1. **Maximality and minimality.** To show that the set of all V-torsion anomalous varieties is non-dense, we only need to consider maximal ones. We recall from the introduction the following definition.

Definition 3.1. *We say that a V-torsion anomalous variety Y is* maximal *if it is not contained in any V-torsion anomalous variety of strictly larger dimension.*

On the other hand, a variety Y can be a component of the intersection of V with different torsion varieties. We want to choose the minimal torsion variety which makes Y anomalous.

Definition 3.2. *Let Y be a V-torsion anomalous variety. We say that the irreducible torsion variety $B + \zeta$ is* minimal *for Y if Y is an irreducible component of $V \cap (B + \zeta)$,*

$$\operatorname{codim} Y < \operatorname{codim} V + \operatorname{codim} B,$$

and $B + \zeta$ has minimal dimension among the irreducible torsion varieties with such properties.

Note that the minimal torsion variety for Y is unique. Indeed if $B+\zeta$ and $B'+\zeta'$ are two torsion varieties which are minimal for Y, then also $(B + \zeta) \cap (B' + \zeta')$ is minimal for Y, and by minimality $\dim((B + \zeta) \cap (B' + \zeta')) = \dim(B + \zeta)$. So $(B' + \zeta') \subseteq (B + \zeta)$. But $(B + \zeta)$ and $(B' + \zeta')$ are irreducible and reduced, thus $(B + \zeta) = (B' + \zeta')$.

3.2. **Relative position of the torsion anomalous varieties.** Without loss of generality we can work with a maximal V-torsion anomalous Y and its minimal torsion variety $B+\zeta$. The maximality for Y avoids redundancy and the minimality assures the weak-transversality of Y in $B + \zeta$, as defined below. The relative position of a V-torsion anomalous variety Y in $B + \zeta$ is determinant and leads to the following natural definition.

Definition 3.3. *An irreducible variety Y is* weak-transverse *in a torsion variety $B + \zeta$ if $Y \subseteq (B + \zeta)$ and Y is not contained in any proper torsion subvariety of $B+\zeta$. Similarly Y is* transverse *in a translate $B+p$ if $Y \subseteq (B+p)$ is not contained in any translate strictly contained in $B+p$. The codimension of Y in $B+\zeta$ is called the* relative codimension of Y in $B + \zeta$*; we simply say the* relative codimension of *Y if Y is V-torsion anomalous and $B + \zeta$ is minimal for Y.*

Then, we have the following lemma.

Lemma 3.4. *Let Y be a maximal V-torsion anomalous variety and let $B + \zeta$ be minimal for Y. Then Y is weak-transverse in $B + \zeta$.*

Proof. Assume that Y is not weak-transverse in $B+\zeta$, then it is contained in an irreducible torsion subvariety $B'+\zeta'$ of $B+\zeta$ with $\operatorname{codim}(B'+\zeta') > \operatorname{codim}(B+\zeta)$. So Y is a component of $V \cap (B'+\zeta')$. In addition

$$\operatorname{codim} Y < \operatorname{codim} V + \operatorname{codim}(B+\zeta) < \operatorname{codim} V + \operatorname{codim}(B'+\zeta'),$$

which contradicts the minimality of $(B+\zeta)$. □

3.3. **Torsion anomalous varieties as components of different intersections.** In the next lemma we prove that every V-torsion anomalous variety which is a component of $V \cap (B+\zeta)$, is also a component of any intersection $V \cap (A+\zeta')$ with $B+\zeta \subseteq A+\zeta'$. Moreover, we can choose $\zeta = \zeta'$, in fact if a translate $B+p$ is contained in another $A+p'$, then $A+p = A+p'$. Indeed B is a subgroup and it contains 0, so $p-p' \in A$ and $A+p' = A+(p-p')+p' = A+p$.

Lemma 3.5. *Let Y be a maximal V-torsion anomalous variety, and let $B+\zeta$ be minimal for Y. Then Y is a component of $V \cap (A+\zeta)$ for every algebraic subgroup $A \supseteq B$, with* $\operatorname{codim} A \geq \dim V - \dim Y$.

Proof. Clearly $Y \subseteq V \cap (A+\zeta)$. Let X be an irreducible component of $V \cap (A+\zeta)$ which contains Y, then

$$\operatorname{codim} X \leq \operatorname{codim} V + \operatorname{codim} A.$$

If the inequality is strict then X is anomalous and by the maximality of Y we get $Y = X$.

Otherwise, since by assumption $\operatorname{codim} A \geq \dim V - \dim Y$, we have

$$\operatorname{codim} X = \operatorname{codim} V + \operatorname{codim} A \geq \operatorname{codim} V + \dim V - \dim Y = \operatorname{codim} Y,$$

but $Y \subseteq X$, so the opposite inequality also holds. This implies $\operatorname{codim} X = \operatorname{codim} Y$. Therefore both varieties are irreducible, so $Y = X$ and Y is a component of $V \cap (A+\zeta)$. □

3.4. **No torsion anomalous varieties on a weak-transverse translate.** The simple choice of maximal and minimal varieties and the group structure allow us to prove Conjecture 1 for weak-transverse translates.

Proof of Theorem 1.1 . Let Y be an $(H+p)$-torsion anomalous variety and let $B+\zeta$ be minimal for Y. Then Y is a component of $(H+p) \cap (B+\zeta)$ and

$$\dim H + \dim B - \dim Y < N.$$

Remark that whenever $(H+p) \cap (B+\zeta) \neq \varnothing$, then $p = h+b+\zeta$ for some $h \in H$ and $b \in B$. Thus

$$(H+p)\cap(B+\zeta) = (H+b+\zeta)\cap(B+\zeta) = (H+b+\zeta)\cap(B+b+\zeta) = (H\cap B)+b+\zeta$$

and

$$\dim(H \cap B) = \dim((H+p) \cap (B+\zeta)).$$

In our case $Y \subseteq (H+p) \cap (B+\zeta)$, then

$$\dim(H \cap B) \geq \dim Y.$$

Thus

$$\dim(H+B) = \dim H + \dim B - \dim(H \cap B) \leq \dim H + \dim B - \dim Y < N,$$

where the last inequality is obtained from the fact that Y is V-torsion anomalous. Therefore $H+B$ is a proper algebraic subgroup of the ambient variety. Since

$p \in H + B + \zeta$, then $H + p \subseteq H + B + \zeta$, against the weak-transversality of $H + p$. $\square$

3.5. **Finitely many maximal V-torsion anomalous varieties in $B+\mathrm{Tor}_G$.** Let V be a weak-transverse variety in G. Let us fix an irreducible torsion subvariety B of G; we end this section by showing the finiteness of the maximal V-torsion anomalous varieties in $V \cap (B + \mathrm{Tor}_G)$ for which $B + \zeta$ is minimal, for some torsion point ζ. This result implies the finiteness of the maximal V-torsion anomalous, if one can uniformly bound the degree of the corresponding minimal torsion variety $B + \zeta$.

We fist prove that the maximal V-torsion anomalous components in $B + \mathrm{Tor}_G$ are non-dense.

Lemma 3.6. *Let V be a weak-transverse variety in G. Let B be an abelian subvariety of G. Then the set of V-torsion anomalous varieties in $V \cap (B + \mathrm{Tor}_G)$ is non-dense in V.*

Proof. Suppose that there exists a dense subset of maximal V-torsion anomalous varieties which are components of $V \cap (B + \zeta)$, for ζ ranging over all torsion points of G. By the box principle, there exists a dense subset of d-dimensional V-torsion anomalous varieties $Y_i \subseteq V \cap (B + \zeta_i)$, for ζ_i torsion points, where d is the maximal integer having this property.

Consider the natural projection $\pi_B : G \to G/B$. As the Y_i are dense and project to torsion points, the dimensional formula tells us that

$$\dim V = \dim Y_i + \dim \pi_B(V).$$

Note that V is not V-torsion anomalous, because V is weak-transverse, thus $\dim Y_i < \dim V$ and $\dim \pi_B(V)$ is at least one. However the Y_i are anomalous therefore

$$\operatorname{codim} Y_i < \operatorname{codim} V + \operatorname{codim} B.$$

We deduce that

$$\dim \pi_B(V) < \operatorname{codim} B = \dim G/B.$$

This shows that $\pi_{B|V}$ is not surjective on G/B. Since V is weak-transverse, also $\pi_B(V)$ is weak-transverse in A/B. Notice that the image via π of any V-torsion anomalous component in $B + \mathrm{Tor}_G$ is a torsion point. By the Manin-Mumford Conjecture the closure of the torsion of $\pi_B(V)$ is non-dense, thus also its preimage is non-dense in V. This contradicts the density of the Y_i. $\square$

An application of this same result actually shows the finiteness of the maximal V-torsion anomalous varieties in $V \cap (B + \mathrm{Tor}_G)$ for which $B + \zeta$ is minimal, for some torsion point ζ.

Lemma 3.7. *Let V be weak-transverse in G. Let B be an abelian subvariety of G. Then there exist only finitely many torsion points ζ such that $V \cap (B + \zeta)$ has a maximal V-torsion anomalous component for which $B + \zeta$ is minimal.*

Proof. Let $\dim G = N$. From Lemma 3.6 the closure of all maximal V-torsion anomalous Y_i which are components of $V \cap (B + \mathrm{Tor}_G)$ is a proper closed subset of V. In particular, if we consider only those for which $B + \zeta$ is minimal, their closure is still a proper closed subset. We write it as the finite union of its irreducible components $X_1 \cup \cdots \cup X_R$. We now show that the X_j are exactly all the maximal V-torsion anomalous components such that $B + \zeta$ is minimal, for some $\zeta \in \mathrm{Tor}_G$.

Suppose that X_1 is not a maximal V-torsion anomalous component for which $B+\zeta$ is minimal, with $\zeta \in \mathrm{Tor}_G$. Then some of the equidimensional Y_i are dense in X_1, $\dim X_1 > \dim Y_i$ and X_1 is not V-torsion anomalous due to the maximality of the Y_i.

By assumption, $B+\zeta_i$ is minimal for Y_i for some $\zeta_i \in \mathrm{Tor}_G$. Then X_1 is not contained in any $B+\zeta$, otherwise $Y_i \subseteq X_1 \subseteq V \cap (B+\zeta)$ and X_1 would be a V-torsion anomalous variety; this gives a contradiction.

Let $H+\zeta$ be the torsion variety (not necessarily proper) of smallest dimension containing X_1. Then X_1 is weak-transverse in $H+\zeta$, and since X_1 is not V-torsion anomalous

$$N - \dim X_1 = N - \dim V + N - \dim H. \tag{1}$$

Recall that, the Y_i are V-torsion anomalous varieties and use (1) to obtain

$$N - \dim Y_i < N - \dim V + N - \dim B = \dim H - \dim X_1 + N - \dim B.$$

whence

$$\dim H - \dim Y_i < \dim H - \dim X_1 + \dim H - \dim B.$$

Since $B+\zeta_i$ is minimal for Y_i, we see that $(B+\zeta_i) \subseteq (H+\zeta)$ and thus $(B+\zeta_i-\zeta) \subseteq H$. Translating every variety by ζ, we obtain that $X_1-\zeta$ is weak-transverse in H and the $Y_i-\zeta$ are dense in $X_1-\zeta$ and $(X_1-\zeta)$-torsion anomalous. This contradicts Lemma 3.6 applied to $X_1-\zeta$ in H. □

4. Main ingredients

4.1. **Notation.** Recall that all varieties are assumed to be defined over the field of algebraic numbers. Let A be an abelian variety; to a symmetric ample line bundle $\mathcal{L}$ on A we attach an embedding $i_{\mathcal{L}} : A \hookrightarrow \mathbb{P}^m$ defined by the minimal power of $\mathcal{L}$ which is very ample. Heights and degrees corresponding to $\mathcal{L}$ are computed via such an embedding. More precisely, the degree of a subvariety of A is the degree of its image under $i_{\mathcal{L}}$; $\hat{h} = \hat{h}_{\mathcal{L}}$ is the $\mathcal{L}$-canonical Néron-Tate height of a point in A, and h is the normalized height of a subvariety of A as defined, for instance, in [Phi91].

Most often we consider products of an elliptic curve E. Then, we denote by $\mathcal{O}_1$ the line bundle on E defined by the neutral element, and by $\mathcal{O}_N$ the bundle on E^N obtained as the tensor product of the pull-backs of $\mathcal{O}_1$ through the N natural projections.

Unless otherwise specified, we compute degrees and heights on E^N with respect to $\mathcal{O}_N$.

We note that by a result of Masser and Wüstholz in [MW93] Lemma 2.2, every abelian subvariety of E^N is defined over a finite extension of k of degree bounded by 3^{16N^4}. For this reason we always assume that all abelian subvarieties are defined over k. Up to a field extension of degree two, we also assume that every endomorphism of E is defined over k.

By $\ll$ we always denote an inequality up to a multiplicative constant depending only on E and N.

4.2. **Subgroups and torsion varieties.** Let $B+\zeta$ be an irreducible torsion variety of E^N with $\operatorname{codim} B = r$. We associate B with a morphism $\varphi_B : E^N \to E^r$ such that $\ker \varphi_B = B + \tau$ with τ a torsion set of absolutely bounded cardinality (by [MW93] Lemma 1.3). In turn φ_B is identified with a matrix in $\mathrm{Mat}_{r\times N}(\mathrm{End}(E))$ of rank r such that the degree of B is essentially (up to constants depending only on N) the sum of the squares of the determinants of the minors of φ_B. By Minkosky's theorem, such a sum is essentially the product of the squares d_i of the norms of the rows of the matrix representing φ_B (see for instance [CVV12] for more details).

In short $B+\zeta$ is a component of the torsion variety given as the zero set of forms $h_1, \dots, h_r$, which are the rows of φ_B, of degree d_i. In addition

$$d_1 \cdots d_r \ll \deg(B+\zeta) \ll d_1 \cdots d_r.$$

We assume to have ordered the h_i by increasing degree.

We also recall that, as is well known, we can use Siegel's lemma to complete the matrix defining B to a square invertible matrix; this gives a construction for the orthogonal complement $B^\perp$ and shows that $\#(B \cap B^\perp) \ll (\deg B)^2$.

As remarked in [Via08], in a product of different elliptic curves an algebraic subgroup is associated with a matrix where the entries corresponding to the non-isogenous factors are all zero.

For this reason our theorems, which we prove in E^N for simplicity, hold in products of different elliptic curves as well.

4.3. **The Zhang Estimate.** We recall the following definition.

Definition 4.1. *For a variety $V \subseteq A$, the* essential minimum $\mu(V)$ *is the supremum of the reals θ such that the set $\{x \in V(\overline{\mathbb{Q}}) \mid \hat{h}(x) \le \theta\}$ is non-dense in V.*

The Bogomolov Conjecture, proved by Ullmo and Zhang in 1998, asserts that the essential minimum $\mu(Y)$ is strictly positive if and only if Y is non-torsion.

From the crucial result in Zhang's proof of the Bogomolov Conjecture (see [Zha98]) and from the definition of normalized height, we have that for an irreducible subvariety X of an abelian variety:

$$\mu(X) \le \frac{h(X)}{\deg X} \le (1+\dim X)\mu(X). \tag{2}$$

4.4. **The Arithmetic Bézout theorem.** The following version of the Arithmetic Bézout theorem is due to Philippon [Phi95].

Theorem 4.2 (Philippon)**.** *Let X and Y be irreducible subvarieties of the projective space $\mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$. Let $Z_1, \dots, Z_g$ be the irreducible components of $X \cap Y$. Then*

$$\sum_{i=1}^{g} h(Z_i) \le \deg X h(Y) + \deg Y h(X) + c(n) \deg X \deg Y,$$

where $c(n)$ is a constant depending only on n.

4.5. **An effective Bogomolov Estimate for relative transverse varieties.** The following theorem is a sharp effective version of the Bogomolov Conjecture for weak-transverse varieties. It is an elliptic analogue, up to a lower order term, of a toric conjecture of Amoroso and David in [AD03].

Theorem 4.3 (Checcoli-Veneziano-Viada). *Let E^N be a product of elliptic curves, and let Y be an irreducible subvariety of E^N transverse in a translate $B+p$. Then, for any $\eta > 0$, there exists a positive constant c_1 depending on E^N and η, such that*

$$\mu(Y) \geq c_1 \frac{(\deg B)^{\frac{1}{\dim B - \dim Y} - \eta}}{(\deg Y)^{\frac{1}{\dim B - \dim Y} + \eta}}.$$

This theorem is a special case of the main theorem of [CVV12].

For our applications, the lower bound must take into account the degree of B. To do so one needs to consider several different line bundles on E^N, and the proof in [CVV12] is based on an equivalence between two line bundles, and on the lower bound for the essential minimum of a transverse variety with respect to the standard bundle $\mathcal{O}_N$. Such a bound is given by Galateau in [Gal10]. The result of Galateau is inspired by the toric version of Amoroso and David [AD03] Theorem 1.4. The constant c_1 is effective and becomes explicit if the constants in [Gal10] are made explicit[1].

4.6. **A relative Lehmer Estimate for points.** In [Car09], Theorem 1.15, Carrizosa proves the so called relative Lehmer problem for CM abelian varieties.

Theorem 4.4 (Carrizosa). *Let A be an abelian defined over a number field k and having CM. Let $\mathcal{L}_0$ be an ample symmetric bundle on A, and let $\mathcal{L} = \mathcal{L}_0^4$. Let H be an abelian subvariety of A of dimension $g_0 > 0$ and P be a point in H, which is not a torsion point modulo all proper abelian subvarieties of H. Then there exists a positive constant $c(A, k, \mathcal{L})$ depending only on A, k and $\mathcal{L}$ such that*

$$\hat{h}_{\mathcal{L}}(P) \geq \frac{c(A,k,\mathcal{L})}{\omega_{k_{\mathrm{tor}}}(P,H)} \left(\frac{\log\log(\omega_{k_{\mathrm{tor}}}(P,H)(\deg_{\mathcal{L}} H)^2)}{\log(\omega_{k_{\mathrm{tor}}}(P,H)(\deg_{\mathcal{L}} H)^2)} \right)^{\kappa(g_0)}$$

where $\kappa(g_0) = 2^{2g_0+1} g_0^{4g_0} (g_0+1)!^{2g_0}$.

Here $k_{\mathrm{tor}} = k(A_{\mathrm{tor}})$ is the field of definition of all the torsion in A and $\omega_{k_{\mathrm{tor}}}(P,H)$ is defined in the following way (see Definition 1.12 of [Car09]):

$$\omega_{k_{\mathrm{tor}}}(P,H) = \min_V \left(\frac{\deg_{\mathcal{L}} V}{\deg_{\mathcal{L}} H} \right)^{1/\mathrm{codim}_H V}$$

where V ranges over all proper subvarieties of H defined over k_{tor}, and containing P.

This theorem generalises to abelian varieties a result of Ratazzi in [Rat04] for one elliptic curve. The proof of Ratazzi is inspired to the theorem of Amoroso and Zannier in [AZ00] for algebraic numbers.

The effectivity in the relative Lehmer is not explicitly stated in the theorem of Carrizosa.[2] Using also the effectivity of other results, as for instance the result of Amoroso and Zannier in [AZ00][3] and of David and Hindry [DH00], one may

[1] In a personal communication Galateau provided us explicit computations.

[2] In a short personal communication she claims that her constants are effective.

[3] Though in [AZ00] the authors are not concerned with effectivity, their result can be made effective (see [AD07]).

check that Carrizosa's constant can be made effective. In addition, the complicated descent in her article can be replaced by the simple induction argument presented for tori in [AV12]. An analogous effective method for the relative Lehmer in tori is given by Delsinne, see [Del09].

As a straightforward corollary of Theorem 4.4 we have the following:

Theorem 4.5. *Let E be an elliptic curve with CM defined over a field k. Let P be a point of infinite order in E^N, and let $B+\zeta$ be the torsion variety of minimal dimension containing P, with B an abelian subvariety and ζ a torsion point. Then for every $\eta > 0$ there exists a positive constant c_2 depending on E^N and η, such that*

$$\hat{h}(P) \geq c_2 \frac{(\deg B)^{\frac{1}{\dim B}-\eta}}{[k_{\mathrm{tor}}(P) : k_{\mathrm{tor}}]^{\frac{1}{\dim B}+\eta}}.$$

We clarify how to obtain this theorem from Theorem 4.4.

We recall that we are assuming all abelian subvarieties to be defined over k,

We remark that $P-\zeta \in B$. In addition $\hat{h}(P) = \hat{h}(P-\zeta)$ and $[k_{\mathrm{tor}}(P) : k_{\mathrm{tor}}] = [k_{\mathrm{tor}}(P-\zeta) : k_{\mathrm{tor}}]$ so we can apply Theorem 4.4 to $P-\zeta$ in B.

Let V be the set of all conjugates of $P-\zeta$ over k_{tor}. Clearly the zero-dimensional variety V is defined over k_{tor}, and it is properly contained in B. Therefore we can say that

$$\omega_{k_{\mathrm{tor}}}(P-\zeta, B) \leq \left(\frac{[k_{\mathrm{tor}}(P) : k_{\mathrm{tor}}]}{\deg_{\mathcal{L}} B}\right)^{1/\dim B}.$$

Notice also that, if we take as $\mathcal{L}$ the fourth power ${\mathcal{O}_N}^4$ of the bundle corresponding to the canonical embedding, this only introduces a constant in the heights and the degrees.

Finally, it is clear that in the statement of Theorem 4.4 the corrective factor can be replaced by

$$\left(\frac{\log\log(\omega_{k_{\mathrm{tor}}}(P,H)(\deg_{\mathcal{L}} H)^2)}{\log(\omega_{k_{\mathrm{tor}}}(P,H)(\deg_{\mathcal{L}} H)^2)}\right)^{\kappa(g_0)} \gg_\eta (\omega_{k_{\mathrm{tor}}}(P,H)(\deg_{\mathcal{L}} H))^{-\eta}.$$

Notice at last that if $B+\zeta$ is the torsion variety of minimal dimension containing P, then $P-\zeta$ is not a torsion point modulo all proper abelian subvarieties of B.

5. Torsion anomalous varieties which are not translates

We let V be a weak-transverse variety in a power of elliptic curves. In this section we prove the finiteness of the maximal V-torsion anomalous varieties which are not translates and have relative codimension one. Note that this theorem holds in any power of elliptic curves, independently if it has or not CM.

Theorem 5.1. *Let $V \subseteq E^N$ be a weak-transverse variety. Then the maximal V-torsion anomalous varieties of relative codimension one which are not translates are finitely many. More precisely, let Y be a maximal V-torsion anomalous variety which is not a translate. Assume that Y has relative codimension one in its minimal $B+\zeta$. Then for any $\eta > 0$ there exist constants depending only on E^N and η such that:*

$$\deg B \ll_\eta (h(V) + \deg V)^{\frac{\operatorname{codim} B}{\operatorname{codim} V - 1}+\eta},$$

$$h(Y) \ll_\eta (h(V) + \deg V)^{\frac{\operatorname{codim} B}{\operatorname{codim} V - 1} + \eta}$$

and

$$\deg Y \ll_\eta \deg V (h(V) + \deg V)^{\frac{\operatorname{codim} B}{\operatorname{codim} V - 1} - 1 + \eta}.$$

In addition the torsion points ζ belong to a finite set.

Proof. Let Y be a maximal V-torsion anomalous variety which is not a translate. Let $B + \zeta$ be minimal for Y. Then Y is a component of $V \cap (B + \zeta)$ with B an abelian variety and ζ a torsion point. In addition $\operatorname{codim} Y < \operatorname{codim} V + \operatorname{codim} B$.

We prove that $\deg B$ is bounded only in terms of V and E^N; we then deduce the bounds for $h(Y)$ and $\deg Y$. By Lemma 3.4 Y is weak-transverse in $B + \zeta$, and by assumption $\operatorname{codim}_{B+\zeta} Y = 1$; therefore Y is transverse in $B + \zeta$. As points are translates and V is not contained in a torsion variety, we have $1 \le \dim Y < \dim V$. Applying the Bogomolov estimate Theorem 4.3 to Y in $B + \zeta$ we get

$$\frac{(\deg B)^{1-\eta}}{(\deg Y)^{1+\eta}} \ll_\eta \mu(Y). \tag{3}$$

We set $r = \operatorname{codim} B$. Let $h_1, \ldots, h_r$ be the forms of increasing degrees d_i such that $B + \zeta$ is a component of their zero set, as recalled in Section 4.2. Then

$$d_1 \cdots d_r \ll \deg(B + \zeta) = \deg B \ll d_1 \cdots d_r. \tag{4}$$

We denote

$$r_1 = \dim V - \dim Y.$$

Note that $r_1 < r$, because Y is V-torsion anomalous. Let A be the algebraic subgroup given by the first $h_1 \cdots h_{r_1}$ forms. Then $\deg A \ll d_1 \cdots d_{r_1}$. Let A_0 be an irreducible component of A containing $B + \zeta$. Then by (4) we have

$$\deg A_0 \ll d_1 \cdots d_{r_1} \ll (\deg B)^{\frac{r_1}{r}},$$

and $\operatorname{codim} A_0 = r_1 = \dim V - \dim Y$. By Lemma 3.5, Y is a component of $V \cap A_0$. We apply the Arithmetic Bézout theorem to $V \cap A_0$ and recall that $h(A_0) = 0$, because A_0 is a torsion variety. Then

$$h(Y) \ll (h(V) + \deg V) \deg A_0 \ll (h(V) + \deg V)(\deg B)^{\frac{r_1}{r}}. \tag{5}$$

For the irreducible variety Y of positive dimension, Zhang's inequality (2) says

$$\mu(Y) \le \frac{h(Y)}{\deg Y}.$$

Combining this with (5) and (3) we obtain

$$\frac{(\deg B)^{1-\eta}}{(\deg Y)^{1+\eta}} \ll_\eta \mu(Y) \ll (h(V) + \deg V) \frac{(\deg B)^{\frac{r_1}{r}}}{\deg Y}.$$

Recall that Y is a component of $V \cap (B + \zeta)$. By Bézout's theorem $\deg Y \le \deg B \deg V$. Thus

$$(\deg B)^{1-\eta} \ll_\eta (h(V) + \deg V)(\deg B)^{\frac{r_1}{r}} (\deg B \deg V)^\eta$$

and therefore

$$(\deg B)^{\frac{r - r_1}{r} - 2\eta} \ll_\eta (h(V) + \deg V)(\deg V)^\eta.$$

Since $r - r_1 = \operatorname{codim} V - 1$, for η small enough we get

$$\deg B \ll_\eta (h(V) + \deg V)^{\frac{r}{\operatorname{codim} V - 1} + \eta} (\deg V)^\eta. \tag{6}$$

So we have proved that the degree of B is bounded only in terms of V and E^N. Since the abelian subvarieties of bounded degree are finitely many, applying Lemma 3.7 we conclude that ζ belongs to a finite set.
Finally, the bound on the height of Y is given by (5) and (6)

$$h(Y) \ll_\eta (h(V) + \deg V)^{\frac{r}{\operatorname{codim} V-1}+\eta}(\deg V)^\eta.$$

The bound on the degree is given by Bézout's theorem for the component Y of $V \cap A_0$ and (6)

$$\deg Y \ll_\eta (h(V) + \deg V)^{\frac{r}{\operatorname{codim} V-1}-1+\eta}(\deg V)^{1+\eta}. \qquad \square$$

6. Torsion anomalous points

In this and the following section we prove that, if V is a weak-transverse variety in a power of elliptic curves with CM, then the V-torsion anomalous varieties which are translates are non-dense in V. We now prove that the V-torsion anomalous varieties of dimension zero are non-dense. The proof relies on the Arithmetic Bézout theorem, the Zhang's inequality and on the relative Lehmer, Theorem 4.5. As the last bound is proved only for CM elliptic curves we need this assumption.

Theorem 6.1. *Let $V \subseteq E^N$ be a weak-transverse variety, where E has CM. Then, the set of maximal V-torsion anomalous points of relative codimension one is a finite set of explicitly bounded height and relative degree.*

More precisely, let k be a field of definition for E and let k_{tor} be the field of definition of all torsion points of E^N. Let d be the dimension of V. Let Y_0 be a maximal V-torsion anomalous point and let $B+\zeta$ be minimal for Y_0, with $\dim B = 1$. *Then*

$$\deg B \ll_\eta ((h(V) + \deg V)[k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}])^{\frac{N-1}{N-1-d}+\eta},$$
$$\hat{h}(Y_0) \ll_\eta (h(V) + \deg V)^{\frac{N-1}{N-1-d}+\eta}[k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}]^{\frac{d}{N-1-d}+\eta},$$
$$[k_{\mathrm{tor}}(Y_0) : k_{\mathrm{tor}}] \ll_\eta \deg V[k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}]^{\frac{N-1}{N-1-d}+\eta}(h(V) + \deg V)^{\frac{d}{N-1-d}+\eta}.$$

In addition the torsion points ζ have order effectively bounded in Theorem 6.2.

Proof. Let Y_0 be a maximal V-torsion anomalous point, with $B+\zeta$ minimal for Y_0.
By assumption $\dim B = \operatorname{codim}_{B+\zeta} Y_0 = 1$. We proceed to bound $\deg B$ and, in turn, the height of Y_0 and its degree over k_{tor}. To this aim we shall use Theorem 4.5 and the Arithmetic Bézout theorem.
By Section 4.2, the variety $B+\zeta$ is a component of the torsion variety defined as the zero set of forms $h_1, \dots, h_{N-1}$ of increasing degrees d_i and

$$d_1 \cdots d_{N-1} \ll \deg B = \deg(B+\zeta) \ll d_1 \cdots d_{N-1}.$$

Consider the torsion variety defined as the zero set of the first d forms $h_1, \dots, h_d$, and take a connected component A_0 containing $B+\zeta$. Then

$$\deg A_0 \ll d_1 \cdots d_d \ll (\deg B)^{\frac{d}{N-1}} \qquad (7)$$

and

$$\operatorname{codim} A_0 = d = \dim V - \dim Y_0.$$

By Lemma 3.5, each component of $V \cap (B+\zeta)$ is a component of $V \cap A_0$. All conjugates of Y_0 over $k_{\mathrm{tor}}(V)$ are in $V \cap (B+\zeta)$, so the number of components of

$V \cap A_0$ is at least

$$[k_{\rm tor}(V, Y_0) : k_{\rm tor}(V)] \geq \frac{[k_{\rm tor}(Y_0) : k_{\rm tor}]}{[k_{\rm tor}(V) : k_{\rm tor}]}.$$

We then apply the Arithmetic Bézout theorem to $Y_0 \subseteq V \cap A_0$ obtaining

$$[k_{\rm tor}(Y_0) : k_{\rm tor}]\hat{h}(Y_0) \ll (h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}](\deg B)^{\frac{d}{N-1}}. \tag{8}$$

Applying Theorem 4.5 to Y_0 in $B + \zeta$, we obtain that for every positive real η

$$\hat{h}(Y_0) \gg_\eta \frac{(\deg B)^{1-\eta}}{[k_{\rm tor}(Y_0) : k_{\rm tor}]^{1+\eta}}. \tag{9}$$

Combining (9) and (8) we have

$$\frac{(\deg B)^{1-\eta}}{[k_{\rm tor}(Y_0) : k_{\rm tor}]^\eta} \ll_\eta [k_{\rm tor}(Y_0) : k_{\rm tor}]\hat{h}(Y_0) \ll$$
$$\ll (h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}](\deg B)^{\frac{d}{N-1}}.$$

For η small enough we obtain

$$\deg B \ll_\eta ((h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}])^{\frac{N-1}{N-1-d}+\eta}[k_{\rm tor}(Y_0) : k_{\rm tor}]^\eta. \tag{10}$$

Apply now Bézout's theorem to $V \cap A_0$. All the conjugates of Y_0 over $k_{\rm tor}(V)$ are components of the intersection, so

$$\frac{[k_{\rm tor}(Y_0) : k_{\rm tor}]}{[k_{\rm tor}(V) : k_{\rm tor}]} \ll_\eta ((h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}])^{\frac{d}{N-1-d}+\eta}(\deg V)^{1+\eta}, \tag{11}$$

which gives the last bound in the statement. Substituting (11) back into (10) we have the bound on $\deg B$.

Finally apply the Arithmetic Bézout theorem to $V \cap A_0$ to get

(12)

$$\hat{h}(Y_0) \ll (h(V){+}\deg V)(\deg B)^{\frac{d}{N-1}} \ll_\eta (h(V){+}\deg V)^{\frac{N-1}{N-1-d}+\eta}[k_{\rm tor}(V) : k_{\rm tor}]^{\frac{d}{N-1-d}+\eta}.$$

Having bounded $\deg B$, in view of Lemma 3.7, the points ζ belong to a finite set. $\square$

Notice that in Theorem 6.1 we have effectively bounded the degree of the abelian variety B, and we applied Lemma 3.7 to prove the finiteness of the points Y_0 in a non effective way. In the following theorem we give a completely effective result. We explicitly bound the degree $[k(Y_0) : \mathbb{Q}]$ for Y_0 not a torsion point; this, together with the bound for $\hat{h}(Y_0)$ in Theorem 6.1, allows to effectively find all V-torsion anomalous points of relative codimension one which are not torsion points. The effectivity of this result is relevant for the applications to the Mordell-Lang Conjecture shown in Section 2.

Theorem 6.2. *Let V be a weak-transverse variety in E^N, where E has CM. Let k be a field of definition for E. Let d be the dimension of V. Let Y_0 be a maximal V-torsion anomalous point which is not a torsion point, and let $B + \zeta$ be minimal for Y_0 with* $\dim B = 1$*. Then*

$$[k(Y_0) : \mathbb{Q}] \ll_\eta ((h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}])^{\frac{d(N-1)}{(N-1-d)^2}+\eta}(\deg V[k(V) : k])^{\frac{N-1}{N-1-d}+\eta}.$$

In addition the torsion points ζ can be chosen with

$$[k(\zeta) : \mathbb{Q}] \ll [k(Y_0) : \mathbb{Q}]$$

and order bounded by

$$\operatorname{ord}(\zeta) \ll_\eta [k(Y_0):\mathbb{Q}]^{\frac{N}{2}+\eta}.$$

Finally let S be the number of maximal V-torsion anomalous points of relative codimension one. Then

$$S \ll_\eta ((h(V)+\deg V)[k_{\mathrm{tor}}(V):k_{\mathrm{tor}}])^{A_1+\eta}(\deg V)^{A_2+1+\eta}[k(V):k]^{A_2+\eta}$$

where

$$A_1 = \frac{(N-1)(2(N+1)(N-d-1)+dN(2N+1))}{2(N-d-1)^2} \leq (N+1)^4,$$
$$A_2 = \frac{N(N-1)(2N+1)}{2(N-d-1)} \leq N^3.$$

Proof. In view of Theorem 6.1, we know that $\deg B$ and $\hat{h}(Y_0)$ are bounded. We now proceed to bound $[k(Y_0):k]$.

To this aim we need to construct an algebraic subgroup G of codimension d defined over k, containing Y_0 and of controlled degree. In order to do this we use Siegel's lemma in a similar way as in [Via03], Proposition 3, which in turn follows the work [BMZ99] of Bombieri, Masser and Zannier in tori. Here we use Siegel's lemma directly on equations with coefficients in the endomorphism ring of E.

We notice that $\operatorname{End}(E)$ is an order in an imaginary quadratic field L with ring of integers $\mathcal{O}$. The coordinates of $Y_0=(x_1,\dots,x_N)$ generate an $\mathcal{O}$-module Γ of rank one. The torsion submodule of Γ is well known; for instance in [Via03], Proposition 2 we find a description for it. Such a torsion module is clearly $\mathcal{O}$ invariant. As a $\mathbb{Z}$-module it is generated by two points $T, \tau T$ of exact orders R, R' respectively, where $R=c(\tau)R'$ and $c(\tau)$ is essentially the real part of τ^2, so a constant of the problem.

Therefore we can write

$$x_i = \alpha_i g + \beta_i T$$

for a fixed point of infinite order g in E, with coefficients $\alpha_i, \beta_i \in \mathcal{O}$ such that

$$\hat{h}(x_i) = |N_L(\alpha_i)|\hat{h}(g) \tag{13}$$

and

$$N_L(\beta_i) \ll R^2.$$

We want to find coefficients $a_i \in \mathcal{O}$ such that $\sum_i^N a_i x_i = 0$. This gives a linear system of 2 equations, obtained equating to zero the coefficients of g and T. The system has coefficients in $\mathcal{O}$ and $N+1$ unknowns: the a_i's and one more unknown for the congruence relation arising from the torsion point.

We use a version of Siegel's lemma over $\mathcal{O}$ as stated in [BG06], Section 2.9, to get d equations with coefficients in $\mathcal{O}$; multiplying them by a constant depending only on E, we may assume that they have coefficients in $\operatorname{End}(E)$. Thus they define the sought-for algebraic subgroup G of degree

$$\deg G \ll \left((\max_i N_L(\alpha_i))(\max_i N_L(\beta_i))\right)^{\frac{d}{N-1}}.$$

Let G_0 be a k-irreducible component of G passing through Y_0. Then

$$\deg G_0 \ll \left((\max_i N_L(\alpha_i))R^2\right)^{\frac{d}{N-1}}.$$

By the maximality of Y_0, the point Y_0 is a component of $V \cap G_0$ and by Bézout's theorem we get

$$\frac{[k(Y_0):k]}{[k(V):k]} \leq \deg V \deg G_0.$$

Using also (13), we obtain

$$\frac{[k(Y_0):k]}{[k(V):k]} \ll \deg V \left((\max_i N_L(\alpha_i))R^2\right)^{\frac{d}{N-1}} \leq \tag{14}$$

$$\leq \deg V \left(R^2 \frac{\max_i \hat{h}(x_i)}{\hat{h}(g)}\right)^{\frac{d}{N-1}}.$$

Notice that $\hat{h}(x_i) \leq \hat{h}(Y_0)$ for all i.

We can now apply Theorem 4.5 to g in E, obtaining for every $\eta > 0$

$$\frac{1}{\hat{h}(g)} \ll_\eta [k_{\rm tor}(g):k_{\rm tor}]^{1+\eta} \leq [k_{\rm tor}(Y_0):k_{\rm tor}]^{1+\eta} \tag{15}$$

because g is defined over $k(Y_0)$.

The product $[k_{\rm tor}(Y_0):k_{\rm tor}]\hat{h}(Y_0)$ was bounded in (8), so using also the bounds in Theorem 6.1 we obtain

$$[k(Y_0):k] \ll_\eta \deg V[k(V):k]((h(V)+\deg V)[k_{\rm tor}(V):k_{\rm tor}])^{\frac{d}{N-1-d}+\eta} R^{2\frac{d}{N-1}}. \tag{16}$$

By a result of Serre in [Ser72], recalled also in [Via03], Corollary 3, for R larger than a constant and φ the Euler function, we have

$$\varphi(R)\varphi(R') \ll [k(Y_0):k].$$

In addition

$$R^{2-\eta} \ll_\eta \varphi(R)\varphi(R')$$

since in general $\varphi(x) \gg_\eta x^{1-\eta}$ and R and R' are related by a constant. From this and (16), for η small enough we obtain

(17)

$$[k(Y_0):k] \ll_\eta \deg V[k(V):k]((h(V)+\deg V)[k_{\rm tor}(V):k_{\rm tor}])^{\frac{d}{N-d-1}+\eta}[k(Y_0):k]^{\frac{d}{N-1}+\eta}.$$

Since Y_0 is V-torsion anomalous of relative dimension one, we have $d < N-1$ and we deduce

$$[k(Y_0):\mathbb{Q}] \ll [k(Y_0):k] \ll_\eta$$

$$\ll_\eta (\deg V[k(V):k])^{\frac{N-1}{N-d-1}+\eta}((h(V)+\deg V)[k_{\rm tor}(V):k_{\rm tor}])^{\frac{d(N-1)}{(N-d-1)^2}+\eta}. \tag{18}$$

We now want to bound the degree of ζ over k and the order of ζ. Let K be the field of definition of $B+\zeta$; we are going to prove that $[k(\zeta):K]$ is absolutely bounded and that $[K:k] \leq [k(Y_0):k]$.

From [Ber87], we can choose ζ in a complement B' of B such that $B \cap B'$ has cardinality bounded only in terms of N. Moreover, we notice that $K \subseteq k(\zeta)$: in fact if $\sigma \in \mathrm{Gal}(\overline{k}/k(\zeta))$, then $\sigma(B+\zeta) = B+\zeta$. Now let $\sigma \in \mathrm{Gal}(\overline{k}/K)$ and suppose that $\sigma(\zeta) \neq \zeta$. Then $\sigma(B+\zeta) = B+\sigma(\zeta) = B+\zeta$, because B is defined over k. Since $\zeta, \sigma(\zeta) \in B'$ we have $\sigma(\zeta) - \zeta \in B \cap B'$. So $[k(\zeta):K] \ll 1$.

We also notice that $K \subseteq k(Y_0)$, otherwise we would have a $\sigma \in \mathrm{Gal}(\overline{k}/k)$ such that $\sigma(Y_0) = Y_0$, but $\sigma(B+\zeta) \neq B+\zeta$. If this were the case, Y_0 would be a component of $V \cap (B+\zeta) \cap \sigma(B+\zeta)$, against the minimality of $B+\zeta$.

Thus

$$[k(\zeta):k]=[k(\zeta):K][K:k]\ll[K:k]\leq[k(Y_0):k].$$

In view of (18), ζ generates an extension of k of bounded degree. By Serre's result mentioned above

$$\operatorname{ord}(\zeta)\ll_\eta [k(Y_0):\mathbb{Q}]^{\frac{N}{2}+\eta}. \tag{19}$$

We are left to give an explicit bound for the number of maximal V-torsion anomalous points Y_0 of relative codimension one. This is obtained in the following way: we first bound the number of possible subgroups B and possible torsion points ζ such that $B+\zeta$ is minimal for some Y_0. Then we apply Bézout's theorem to every intersection $V\cap(B+\zeta)$.

We already proved in Theorem 6.1 and (19) that if $B+\zeta$ is minimal for Y_0, then $\deg B$ and $\operatorname{ord}(\zeta)$ are bounded.

By Section 4.2, the number of abelian subvarieties B in E^N of dimension one and degree at most $\deg B$ is $\ll_\eta (\deg B)^{N+\eta}$, for every $\eta>0$. In fact, if B is such a abelian subvariety, consider its associated matrix, as recalled in Section 4.2; it is an $(N-1)\times N$ matrix and we call d_i the square of the norm of its i-th row. We let $\deg B={p_1}^{e_1}\cdots{p_r}^{e_r}$ be the factorization of $\deg B$ into distinct prime factors. Then the number of possible choices for the elements d_i is bounded by $\delta(\deg B)^{N-1}$, where, for a positive integer n, $\delta(n)$ counts the number of divisors of n. We notice that, for every $\eta>0$ we have

$$\delta(\deg B)\ll_\eta(\deg B)^\eta.$$

Now, for every choice of the d_i, the number D of $(N-1)\times N$ matrices in which the square of the norm of the i-th row is at most d_i is bounded in the following way

$$D\ll\left(\prod_{i=1}^{N-1}d_i\right)^N\ll(\deg B)^N.$$

So for every $\eta>0$, the number of possible subgroups B is $\ll_\eta(\deg B)^{N+\eta}$.

As for the point ζ, it is well known that the number of torsion points in E^N of order bounded by a constant M is at most M^{2N+1}. In fact the number of points of order dividing a positive integer i is i^{2N}; so a bound for the number of torsion points of order at most M is given by

$$\sum_{i=1}^M i^{2N}\ll M^{2N+1}.$$

Applying Bézout's theorem to every intersection $V\cap(B+\zeta)$, we obtain that for every $\eta>0$ the number S of V-torsion anomalous points of relative codimension one is bounded by

$$S\ll_\eta\deg V(\deg B)^{N+1+\eta}\operatorname{ord}(\zeta)^{2N+1}.$$

This, combined with Theorem 6.1, (19) and (18), gives the required explicit bounds. $\square$

7. Torsion anomalous translates of positive dimension

We let V be a weak-transverse variety in a power of elliptic curves with CM. In this section we study V-torsion anomalous varieties which are translates of positive dimension. We reduce this case to the zero dimensional case.

First we compare the V-torsion anomalous translates with translates contained in V.

Lemma 7.1. *Let V be a weak-transverse subvariety of an abelian variety of dimension N. Let Y be a maximal V-torsion anomalous translate, then Y is a maximal translate contained in V (i.e. Y is not strictly contained in any translate contained in V).*

Proof. Let Y be a maximal V-torsion anomalous translate and suppose that Y is contained in a maximal translate $(H+p) \subseteq V$ with $\dim(H+p) > \dim Y$. Let $B+\zeta$ be minimal for Y. Then Y is a component of $V \cap (B+\zeta)$ and

$$\operatorname{codim} Y < \operatorname{codim} V + \operatorname{codim} B. \tag{20}$$

Since $(H+p) \cap (B+\zeta) \supset Y$ then $p = h + b + \zeta$ for some $h \in H$ and $b \in B$. Therefore

$$(H+p)\cap(B+\zeta) = (H+b+\zeta)\cap(B+\zeta) = (H+b+\zeta)\cap(B+b+\zeta) = (H\cap B)+b+\zeta$$

and

$$\dim(H \cap B) = \dim((H+p) \cap (B+\zeta)) \geq \dim Y.$$

By (20), we deduce

$$\begin{aligned} \dim(H+B) &= \dim H + \dim B - \dim(H \cap B) \\ &\leq \dim H + \dim B - \dim Y < N. \end{aligned} \tag{21}$$

So $H + B + \zeta$ is a proper torsion subvariety of the ambient variety. Moreover

$$H + p = H + h + b + \zeta \subseteq H + B + \zeta.$$

Thus

$$H + p \subseteq V \cap (H + B + \zeta).$$

By (20) and (21), we deduce

$$\begin{aligned} N - \dim H &< N - \dim V + N - \dim H - \dim B + \dim Y \\ &\leq N - \dim V + N - \dim H - \dim B + \dim(B \cap H) \\ &= N - \dim V + N - \dim(H + B). \end{aligned}$$

This implies that $H+p$ is a V-torsion anomalous translate, against the maximality of Y. $\square$

The following lemma is a straightforward corollary of the result proved by Patrice Philippon in the Appendix A. It relates the essential minimum of a translate to the height of the point of translation.

Lemma 7.2. *Let $H + Y_0$ be a weak-transverse translate in E^N, with Y_0 a point in the orthogonal complement $H^\perp$ of H. Then*

$$\mu(Y_0) = \mu(H + Y_0).$$

Proof. The points $Y_0+\zeta$, for $\zeta\in\mathrm{Tor}_H$, are dense in $H+Y_0$ and they have height equal to $\hat{h}(Y_0)$. So we get $\mu(H+Y_0)\le\mu(Y_0)$.

To obtain the other inequality, consider a set of points of the form x_i+Y_0 with $x_i\in H$, which is dense in $H+Y_0$. By the Lemma of Philippon in the Appendix A, we get $\hat{h}(x_i+Y_0)=\hat{h}(x_i)+\hat{h}(Y_0)\le\hat{h}(Y_0)$. □

We now state our main theorem for V-torsion anomalous translates. Let $H+Y_0$ be a maximal V-torsion anomalous translate of relative codimension one. The idea is to apply the functorial Lehmer-type bound by Carrizosa to the point Y_0 in the complement $H^\perp$ of H, so that the problem becomes zero dimensional. We then apply the Arithmetic Bézout theorem to $H+Y_0$ in the usual way. The link between $\mu(Y_0)$ and $\mu(H+Y_0)$ is then given by Lemma 7.2.

Theorem 7.3. *Let V be a weak-transverse subvariety of E^N, where E has CM. Then the set of V-torsion anomalous translates of relative codimension one is a finite set of explicitly bounded normalized height and degree.*

More precisely, let k be a field of definition for E and let k_{tor} be the field of definition of all torsion points of E^N. Let $H+p$ be a maximal V-torsion anomalous translate of relative codimension one. Let $B+\zeta$ be minimal for $H+p$. Then, for every real positive η there exist constants depending only on E^N and η, such that

$$\begin{aligned}\deg B &\ll_\eta ((h(V)+\deg V)[k_{\mathrm{tor}}(V):k_{\mathrm{tor}}])^{\frac{\operatorname{codim} B}{\operatorname{codim} V-1}+\eta},\\ h(H+p) &\ll_\eta (h(V)+\deg V)^{\frac{\operatorname{codim} B}{\operatorname{codim} V-1}+\eta}[k_{\mathrm{tor}}(V):k_{\mathrm{tor}}]^{\frac{\dim V-\dim B+1}{\operatorname{codim} V-1}+\eta},\\ \deg(H+p) &\ll_\eta (\deg V)((h(V)+\deg V)[k_{\mathrm{tor}}(V):k_{\mathrm{tor}}])^{\frac{\dim V-\dim B+1}{\operatorname{codim} V-1}+\eta}.\end{aligned}$$

In addition the points ζ belong to a finite set (of cardinality absolutely bounded in Theorem 7.5).

Proof. As we remarked in Section 4.6, we can assume all abelian subvarieties of E^N to be defined over k.

Let Y_0 be a point in the orthogonal complement $H^\perp$ of H, such that $H+p=H+Y_0$. By Lemma 7.2,

$$\mu(Y_0)=\mu(H+Y_0). \tag{22}$$

We are going to use the Arithmetic Bézout theorem to find an upper bound for $\mu(H+Y_0)$ and Theorem 4.5 to find a lower bound.

Let $B+\zeta$ be minimal for $H+Y_0$, with ζ in the orthogonal complement of B. Then $H+Y_0$ is a component of $V\cap(B+\zeta)$ and by assumption $\operatorname{codim}_{B+\zeta}(H+Y_0)=1$. By Lemma 3.4 $H+Y_0$ is weak-transverse in $B+\zeta$, thus Y_0 is not a torsion point and the coordinates of Y_0 generate a module of rank $\operatorname{codim}_{B+\zeta}(H+Y_0)=1$.

Let r be the codimension of B. The variety $B+\zeta$ is a component of the torsion variety given by the zero set of the forms $h_1,\dots,h_r$ of increasing degrees d_i and

$$d_1\cdots d_r\ll\deg B=\deg(B+\zeta)\ll d_1\cdots d_r.$$

Note that $H+Y_0$ is contained in V and, by assumption, it has relative codimension one. Thus $\operatorname{codim} V<\operatorname{codim} H=r+1$. Denote

$$r_1=r+1-\operatorname{codim} V=\dim V-\dim H.$$

Consider the torsion variety defined as the zero set of the first r_1 forms $h_1, \dots, h_{r_1}$, and take one of its irreducible component A_0 passing through $H+Y_0$. Then

$$\deg A_0 \ll (\deg B)^{\frac{r_1}{r}} \tag{23}$$

and

$$\operatorname{codim} A_0 = r_1.$$

Recall that k_{tor} is the field of definition of all torsion points in E^N. By Lemma 3.5, $H+Y_0$ is a component of $V \cap A_0$. Since the intersection $V \cap A_0$ is defined over k_{tor}, every conjugate of $H+Y_0$ over k_{tor} is a component of $V \cap A_0$ and all such components have the same normalized height.

All conjugates of $H+Y_0$ over $k_{\text{tor}}(V)$ are in $V \cap (B+\zeta)$. So the number of components of $V \cap A_0$ is at least

$$\frac{[k_{\text{tor}}(H+Y_0) : k_{\text{tor}}]}{[k_{\text{tor}}(V) : k_{\text{tor}}]}.$$

We apply the Arithmetic Bézout theorem to $H+Y_0$ in $V \cap A_0$ and we obtain

$$h(H+Y_0) \ll (h(V) + \deg V)(\deg B)^{\frac{r_1}{r}} \tag{24}$$

and estimating the components we have

$$h(H+Y_0)\frac{[k_{\text{tor}}(H+Y_0) : k_{\text{tor}}]}{[k_{\text{tor}}(V) : k_{\text{tor}}]} \ll (h(V) + \deg V)(\deg B)^{\frac{r_1}{r}}. \tag{25}$$

By the Zhang's inequality, (22) and (25), we deduce

$$\mu(Y_0) \ll \frac{(h(V) + \deg V)[k_{\text{tor}}(V) : k_{\text{tor}}](\deg B)^{\frac{r_1}{r}}}{[k_{\text{tor}}(H+Y_0) : k_{\text{tor}}] \deg H}. \tag{26}$$

Consider the smallest abelian subvariety H' of B containing $Y_0 - \zeta$, so Y_0 is not contained in any torsion subvariety of $H' + \zeta$. The relative codimension of $H + Y_0$ in $B + \zeta$ is one therefore the dimension of H' is one. Moreover $H^\perp \cap (B+\zeta)$ has dimension one and it contains Y_0. Consider the irreducible component $H_0^\perp$ of the intersection containing Y_0: since Y_0 is not torsion, then $H_0^\perp$ has dimension one. So $H' + \zeta = H_0^\perp$, because both varieties are irreducible, contain Y_0 and are one dimensional. Notice in particular that $\zeta \in H_0^\perp \subseteq H^\perp$, and therefore $H' \subseteq H^\perp$.

We also notice that

$$[k_{\text{tor}}(Y_0) : k_{\text{tor}}] \leq [k_{\text{tor}}(H+Y_0) : k_{\text{tor}}] \cdot \#(H \cap H'). \tag{27}$$

In fact if $\sigma \in \operatorname{Gal}(\overline{k_{\text{tor}}}/k_{\text{tor}}(H+Y_0))$ then $\sigma(Y_0) - Y_0 \in H$. Since $H' + \zeta = H' + Y_0$ and it is defined over k_{tor}, we have that $\sigma(Y_0) - Y_0 \in H'$ as well. Hence the number of conjugates of Y_0 over k_{tor} is at most $\#(H \cap H')$.

For the lower bound for $\mu(Y_0)$, the proof follows the case of dimension zero. In particular applying Theorem 4.5 to Y_0 in $H' + \zeta$ we get that, for every positive real η

$$\mu(Y_0) = \hat{h}(Y_0) \gg_\eta \frac{(\deg H')^{1-\eta}}{[k_{\text{tor}}(Y_0) : k_{\text{tor}}]^{1+\eta}}. \tag{28}$$

Combining (26), (27) and (28) we get

$$(\deg H')^{1-\eta} \ll_\eta (h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}] \frac{(\deg B)^{\frac{r_1}{r}}}{\deg H} \cdot \\ \cdot \#(H \cap H')^{1+\eta} [k_{\rm tor}(H + Y_0) : k_{\rm tor}]^\eta. \tag{29}$$

Notice that, by the definition of H', $B = H + H'$ and

$$\#(H \cap H') \deg B = \deg H \deg H'. \tag{30}$$

Thus, possibly changing η, we have

$$\deg B \ll_\eta ((h(V)+\deg V)[k_{\rm tor}(V) : k_{\rm tor}])^{\frac{r}{\operatorname{codim} V - 1}+\eta} (\#(H \cap H')[k_{\rm tor}(H + Y_0) : k_{\rm tor}])^\eta . \tag{31}$$

We now want to remove the dependence on $\#(H \cap H')$ and $[k_{\rm tor}(H + Y_0) : k_{\rm tor}]$ and bound the degree of the translate $H + Y_0$.

We first apply Bézout's theorem to the intersection $V \cap A_0$, obtaining:

$$\deg H \leq \deg V \deg A_0 \ll \deg V (\deg B)^{\frac{r_1}{r}} \tag{32}$$

and estimating the components we obtain

$$\deg H \frac{[k_{\rm tor}(H + Y_0) : k_{\rm tor}]}{[k_{\rm tor}(V) : k_{\rm tor}]} \leq \deg V \deg A_0 \ll \deg V (\deg B)^{\frac{r_1}{r}}. \tag{33}$$

We remarked that $H' \subseteq H^\perp$, therefore $H \cap H' \subseteq H \cap H^\perp$ and,

$$\#(H \cap H') \leq \#(H \cap H^\perp) \ll (\deg H)^2.$$

This and (33) give

$$\#(H \cap H')[k_{\rm tor}(H + Y_0) : k_{\rm tor}] \ll ([k_{\rm tor}(V) : k_{\rm tor}] \deg V)^2 (\deg B)^{\frac{2r_1}{r}}.$$

Substituting in (31) we get

$$\deg B \ll_\eta ((h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}])^{\frac{r}{\operatorname{codim} V - 1}+\eta}. \tag{34}$$

Then, using (32) and replacing (34) we have

$$\deg(H + Y_0) \ll (\deg V)(\deg B)^{\frac{r_1}{r}} \ll_\eta \\ \ll_\eta (\deg V)((h(V) + \deg V)[k_{\rm tor}(V) : k_{\rm tor}])^{\frac{r_1}{\operatorname{codim} V - 1}+\eta}.$$

Finally, from (24) and (34) we obtain

$$h(H + Y_0) \ll_\eta (h(V) + \deg V)^{\frac{r}{\operatorname{codim} V - 1}+\eta} [k_{\rm tor}(V) : k_{\rm tor}]^{\frac{r_1}{\operatorname{codim} V - 1}+\eta}.$$

Since we have bounded $\deg B$, we can conclude from Lemma 3.7 that the points ζ belong to a finite set. $\square$

In the proof we bounded the degree of $H + Y_0$ using Bézout's theorem, and we obtained a bound depending on $\deg V$ and $h(V)$. The dependence on $h(V)$ may be removed with a different argument.

Let $H+p \subseteq V$ be a translate which is maximal with respect to the inclusion among all such translates. Bombieri and Zannier in [BZ96], Lemma 2, proved that only finitely many such abelian subvarieties H can occur. More precisely the maximal translates contained in V have degree bounded only in terms of the degree and the dimension of V. As a corollary of their proof we obtain the following lemma.

Lemma 7.4 (Bombieri-Zannier)**.** *Let V be a weak-transverse variety. Then the maximal translates contained in V are of the form $H+p$ for finitely many abelian subvarieties H with*

$$\deg H \leq (\deg V)^{2^{\dim V}}.$$

Using this lemma we obtain a bound which is more uniform, though the dependence on $\deg V$ shows a bigger exponent.

As remarked for the zero dimensional case, in Theorem 7.3 we proved finiteness using the non effective Lemma 3.7. We now give a completely effective result which is the analogous of Theorem 6.2 in positive dimension. We bound the degrees of the fields of definition of the translates $H+p$ and of the torsion points ζ.

Theorem 7.5. *Let V be a weak-transverse subvariety of E^N, where E has CM. Let k be a field of definition for E. Let $H+p$ be a maximal V-torsion anomalous translate of relative codimension one, and let $B+\zeta$ be minimal for $H+p$. Set $r = \operatorname{codim} B$; then the field $k(H+p)$ of definition of $H+p$ has degree bounded by*

$$[k(H+p):\mathbb{Q}] \ll_\eta [k(V):k]^{r+\eta} (\deg V)^{3r-1} \cdot$$
$$\cdot\, (h(V)+\deg V)^{\frac{(2r-1)(r-\operatorname{codim} V+1)+r(r-1)}{\operatorname{codim} V-1}+\eta} [k_{\mathrm{tor}}(V):k_{\mathrm{tor}}]^{\frac{(3r-2)(r-\operatorname{codim} V+1)}{\operatorname{codim} V-1}}.$$

Moreover the torsion points ζ can be chosen so that

$$[k(\zeta):\mathbb{Q}] \ll_\eta [k(H+p):\mathbb{Q}]$$

and

$$\operatorname{ord}(\zeta) \ll_\eta [k(H+p):\mathbb{Q}]^{\frac{N}{2}+\eta}.$$

Proof. We keep all the notations and definitions used in Theorem 7.3.

Of course

$$[k(H+p):k] = [k(H+Y_0):k] \leq [k(Y_0):k],$$

because we are assuming all abelian subvarieties of E^N to be defined over k.

The bound on the degree $[k(Y_0):k]$ is obtained following the same idea of Theorem 6.2: since H' has dimension one, the group generated by the coordinates of Y_0 is an $\operatorname{End}(E)$-module of rank one. We can apply Siegel's lemma in a similar way to the zero-dimensional case, where we use the estimate (26) for the height of Y_0. In this case, however, we want to find an algebraic subgroup G of dimension 2, containing Y_0, and contained in $H^\perp$.

We know that $\operatorname{End}(E)$ is an order in an imaginary quadratic field L. Let $\lambda_1, \ldots, \lambda_{N-r-1}$ be linear forms which give equations for $H^\perp$, and define

$$\delta_i = \max_j |N_L(l_{ij})|,$$

where $(l_{ij})_j$ is the vector of coefficients of the form λ_i. We now follow the steps of the proof of Theorem 6.2 and apply Siegel's lemma to obtain $r-1$ independent solutions which are also orthogonal to the vectors of coefficients of $\lambda_1, \ldots, \lambda_{N-r-1}$; this time, in addition to the two equations of Theorem 6.1, we have also one equation for each of the λ_i, for a total of $N-r+1$ equations in $N+1$ unknowns (N coefficients and one for the torsion point).

These $r-1$ vectors give $r-1$ linear forms which, together with the $\lambda_1, \ldots, \lambda_{N-r-1}$, provide the $N-2$ equations needed to define G.

The bounds provided by Siegel's lemma give

$$\deg G \ll_\eta \left(\prod_{i=1}^{N-r-1} \delta_i\right)\left(\hat{h}(Y_0)[k(Y_0):k]^{1+\eta}\prod_{i=1}^{N-r-1}\delta_i\right)^{\frac{r-1}{r}} \ll_\eta \tag{35}$$
$$\ll_\eta (\deg H)^{2-\frac{1}{r}}\hat{h}(Y_0)^{\frac{r-1}{r}}[k(Y_0):k]^{\frac{r-1}{r}+\eta}.$$

Now that we have found G, we first show that $H+Y_0$ is a component of $V\cap(H+G)$: indeed $\dim(H+G)=\dim B+1$, because $G\subseteq H^\perp$ has dimension 2; therefore any component of the intersection $V\cap(H+G)$ with dimension greater than $\dim H$ is anomalous. But $H+Y_0$ is clearly contained in $V\cap(H+G)$ and, by the maximality of $H+Y_0$, it is not contained in any V-anomalous subvariety of greater dimension; hence it is itself a component of $V\cap(H+G)$ and so are all its conjugates over $k(V)$.

Applying Bézout's theorem to $V\cap(H+G)$ we get

$$\frac{[k(Y_0):k]}{[k(V):k]}\deg H \le \deg V \deg(H+G) = \deg V \deg H \deg G,$$
$$[k(Y_0):k] \le [k(V):k]\deg V \deg G, \tag{36}$$

and this gives the desired bound.

Substituting (35) in (36), and using all the bounds from Theorem 7.3 we obtain

$$[k(Y_0):\mathbb{Q}] \ll_\eta [k(V):k]^{r+\eta}(\deg V)^{3r-1}(h(V)+\deg V)^{\frac{(2r-1)r_1+r(r-1)}{\operatorname{codim} V-1}+\eta}\cdot$$
$$\cdot[k_{\mathrm{tor}}(V):k_{\mathrm{tor}}]^{\frac{(3r-2)r_1}{\operatorname{codim} V-1}}.$$

Finally, as in the proof of Theorem 6.1, ζ can be chosen so that $[k(\zeta):\mathbb{Q}] \ll [k(Y_0):\mathbb{Q}]$; again by Serre's theorem, the points ζ have order bounded by $\operatorname{ord}(\zeta)\ll_\eta [k(Y_0):\mathbb{Q}]^{\frac{N}{2}+\eta}$ and therefore belong to an explicit finite set.

Finally, the set of all possible translates $H+p$ has cardinality S bounded by

$$S \ll_\eta [k(V):k]^{D_1}(\deg V)^{D_2}(h(V)+\deg V)^{D_3}[k_{\mathrm{tor}}(V):k_{\mathrm{tor}}]^{D_4} \tag{37}$$

where

$$D_1 = \frac{rN(2N+1)}{2}$$
$$D_2 = \frac{(3r-1)N(2N+1)}{2}+1$$
$$D_3 = \frac{(N+1)r}{\operatorname{codim} V-1} + \frac{N(2N+1)}{2}\frac{(2r-1)(r-\operatorname{codim} V+1)+r(r-1)}{\operatorname{codim} V-1}$$
$$D_4 = \frac{(N+1)r}{\operatorname{codim} V-1} + \frac{N(2N+1)}{2}\frac{(3r-2)(r-\operatorname{codim} V+1)}{\operatorname{codim} V-1}.$$

□

8. Proof of Theorem 1.3

We now give the proof of Theorem 1.3, restated here for clarity.

Theorem 8.1. *Let V be a weak-transverse variety in an abelian variety A with CM. Let g be the maximal dimension of a simple factor of A. If $\operatorname{codim} V \ge g+1$, then the maximal V-torsion anomalous varieties of relative codimension 1 are finitely many and they have degree and normalised height effectively bounded.*

Proof. Let X be a maximal V-torsion anomalous variety of relative codimension 1 and let $B+\zeta$ be minimal for X. Then X is a component of the intersection $V\cap(B+\zeta)$ and $\operatorname{codim} V > \dim B - \dim X = 1$.

Since $\operatorname{codim} V \geq g+1$ by the hypothesis and $\dim V \geq \dim X + 1 = \dim B$, then $\operatorname{codim} B \geq g+1$. From Section 4.2, $B+\zeta$ is a component of the torsion variety given by the zero set of $r \geq 2$ forms $h_1, \dots, h_r$ of increasing degrees d_i and

$$d_1 \cdots d_r \ll \deg(B+\zeta) \ll d_1 \cdots d_r.$$

Notice that one single equation gives $\operatorname{codim} B \leq g$, since g is the maximal dimension of a simple factor of A. This implies that $r \geq 2$.

Denote by A_0 the algebraic subgroup given by $h_1, \dots, h_{r-1}$. Since $r \geq 2$, we have $A_0 \subsetneq A$. In addition $\deg A_0 \ll (\deg B)^{\frac{r-1}{r}}$.

We now show that X is a component of $V \cap A_0$. Suppose, to the contrary, that $X \subsetneq Y \subseteq V \cap A_0$ where Y is a component of $V \cap A_0$ strictly containing X. By maximality of X, Y cannot be V-torsion anomalous, hence

$$\operatorname{codim} Y = \operatorname{codim} V + \operatorname{codim} A_0.$$

Moreover by construction $\dim A_0 \leq \dim B + g$. This implies

$$\dim B - 1 = \dim X < \dim Y = \dim A_0 - \operatorname{codim} V \leq \dim B + g - \operatorname{codim} V$$

and so $\operatorname{codim} V < g+1$, which contradicts the hypothesis on V.

So X is a component of $V \cap A_0$. The bounds for the normalised height and for the degree of X are now obtained following the proof in the case of elliptic curves. Namely we combine the Zhang inequality, the Arithmetic Bézout theorem, the Bogomolov estimate in Theorem 4.3 and the relative Lehmer estimate in Theorem 4.5. The appendix is here replaced by the general result of Bertrand in [Ber86].

If X is not a translate, the same argument of Theorem 5.1 applies to show that for any positive real η, there exist constants depending only on A and η such that

$$h(X) \ll_\eta (h(V) + \deg V)^{\operatorname{codim} B + \eta} (\deg V)^{\eta} \tag{38}$$

and

$$\deg X \ll_\eta (h(V) + \deg V)^{\operatorname{codim} B - 1 + \eta} (\deg V)^{1+\eta}. \tag{39}$$

If X is a translate, we proceed as in the proofs of Theorem 6.1 if $\dim X = 0$ and of Theorem 7.3 if $\dim X \geq 1$, obtaining that for any positive real η, there exist constants depending only on A and η such that

$$h(X) \ll_\eta (h(V) + \deg V)^{\operatorname{codim} B + \eta} [k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}]^{\operatorname{codim} B - 1 + \eta} \tag{40}$$

and

$$\deg X \ll_\eta \deg V ((h(V) + \deg V)[k_{\mathrm{tor}}(V) : k_{\mathrm{tor}}])^{\operatorname{codim} B - 1 + \eta}. \tag{41}$$

Here k is the field of definition for A, k_{tor} is the field of definiton for the torsion points of A and $\deg X = [k_{\mathrm{tor}}(X) : k_{\mathrm{tor}}]$ when X is a point. □

9. The case of a curve

In this section we extend our method in order to get some further results for curves.

Theorem 9.1. *Let C be a weak-transverse curve in E^N, where E has CM. Let k be a field of definition for E and denote by $k_{\rm tor}$ the field of definition of all torsion points of E^N. Then the set*

$$\mathcal{S}(C) = C \cap (\cup_{\operatorname{codim} H > \dim H} H)$$

is a finite set of effectively bounded Néron-Tate height. Here H ranges over all algebraic subgroups of codimension larger than the dimension.

More precisely the union can be taken over finitely many algebraic subgroups

$$\mathcal{S}(C) = C \cap \bigcup_{i=1}^{M} H_i;$$

the H_i are algebraic subgroups with $\dim H < \operatorname{codim} H$, and for any real $\eta > 0$, there exist constants depending only on E^N and η such that

$$\deg H_i \ll_\eta ((h(C)+\deg C)[k_{\rm tor}(C):k_{\rm tor}])^{\frac{r(N-r)(N+2r-2)}{2(r-1)(2r-N)}+\eta}([k(C):k]\deg C)^{\frac{Nr}{2(r-1)}+\eta},$$

where $r = \operatorname{codim} H_i$. Moreover M can be effectively bounded.

Furthermore, if $Y_0 \in \mathcal{S}(C)$ we have

$$\hat{h}(Y_0) \ll_\eta (h(C)+\deg C)^{\frac{N+1}{2}+\eta}[k_{\rm tor}(C):k_{\rm tor}]^{\frac{N-1}{2}+\eta};$$

if Y_0 is not a torsion point we also have

$$[k(Y_0):\mathbb{Q}] \ll_\eta ([k(C):k]\deg C)^{\frac{N+1}{N-1}+\eta}((h(C)+\deg C)[k_{\rm tor}(C):k_{\rm tor}])^{\frac{N+1}{2}+\eta},$$

and the cardinality of the points in $\mathcal{S}(C)$ which are not torsion points is bounded as

$$S \ll_\eta [k(C):k]^{B_1}(\deg C)^{B_1+1+\eta}((h(C)+\deg C)[k_{\rm tor}(C):k_{\rm tor}])^{B_2+\eta},$$

where

$$B_1 = \frac{N(N+1)(2N+1)}{2(N-1)}, \qquad B_2 = \frac{(3N^2+N-1)(N+1)}{4}.$$

Proof. We notice that the set of all torsion points in C is finite and has cardinality bounded by the effective Manin-Mumford Conjecture. From now on, we will be concerned with points in $\mathcal{S}(C)$ that are not torsion.

Clearly all the points in the intersection $C \cap (\cup_{\operatorname{codim} H > \dim H} H)$ are C-torsion anomalous. In addition since C is a weak-transverse curve each torsion anomalous point is maximal.

Let $Y_0 \in \mathcal{S}(C)$ be a non-torsion point; then $Y_0 \in C \cap H$, with H the minimal subgroup containing Y_0 with respect to the inclusion, and let $r = \operatorname{codim} H$.

Let also $B+\zeta$ be a component of H containing Y_0. Clearly $\dim B = \dim H$ and $Y_0 \in C \cap (B+\zeta)$ with $B+\zeta$ minimal for Y_0; as done several times, we can also choose the torsion point ζ in the orthogonal complement of B.

Notice that

$$\deg H \leq (\deg B)\operatorname{ord}(\zeta). \tag{42}$$

In fact $B + \langle\zeta\rangle$ is an algebraic subgroup of dimension equal to $\dim H$, it contains Y_0 and it is contained in H; thus $B + \langle\zeta\rangle = H$, by the minimality of H.

The proof now follows the lines of the proofs of Theorems 6.1 and 6.2. We proceed to bound $\deg B$ and, in turn, the height of Y_0, using Theorem 4.5 and the Arithmetic Bézout theorem. Then, using Siegel's lemma, we get a bound for $[k(Y_0):k]$ and for the order and the number of torsion points ζ, providing also a bound for $\deg H$.

Recall that we have $r = \operatorname{codim} B = \operatorname{codim} H$.

We first exclude the case $r = 1$ and show that the case $N - r = 1$ is covered by Theorems 6.1 and 6.2. If $r = 1$ then $\dim B < \operatorname{codim} B$ implies $\dim B = 0$ and $N = \dim B + \operatorname{codim} B = 1$ contradicting the weak-transversality of C.

The case $N - r = 1$ corresponds to $\dim B = 1$ and Y_0 of relative codimension one, that can be treated with Theorems 6.1 and 6.2.

Thus we can assume $r > N - r \geq 2$. Moreover $2r > N$, since by assumption $\operatorname{codim} B > \dim B$.

As done several times, by Section 4.2, the variety $B + \zeta$ is a component of the zero set of forms $h_1, \ldots, h_r$ of increasing degrees d_i and

$$d_1 \cdots d_r \ll \deg B = \deg(B + \zeta) \ll d_1 \cdots d_r.$$

Consider the torsion variety defined as the zero set of h_1, and let A_0 be one of its connected components containing $B + \zeta$.

Then

$$\deg A_0 \ll d_1 \ll (\deg B)^{\frac{1}{r}}. \tag{43}$$

From Theorem 4.5 applied to Y_0 in $B + \zeta$, for every positive real η we get

$$\hat{h}(Y_0) \gg_\eta \frac{(\deg B)^{\frac{1}{N-r} - \eta}}{[k_{\text{tor}}(Y_0) : k_{\text{tor}}]^{\frac{1}{N-r} + \eta}}. \tag{44}$$

Notice that all conjugates of Y_0 over $k_{\text{tor}}(C)$ are components of $C \cap A_0$. Applying the Arithmetic Bézout theorem to Y_0 in $C \cap A_0$ and arguing as in the proof of Theorem 6.1, we have

$$\frac{[k_{\text{tor}}(Y_0) : k_{\text{tor}}]}{[k_{\text{tor}}(C) : k_{\text{tor}}]} \hat{h}(Y_0) \ll (h(C) + \deg C)(\deg B)^{\frac{1}{r}}. \tag{45}$$

From (44) and (45) we get

$$(\deg B)^{\frac{2r-N}{r(N-r)} - \eta} \ll_\eta (h(C) + \deg C)[k_{\text{tor}}(C) : k_{\text{tor}}][k_{\text{tor}}(Y_0) : k_{\text{tor}}]^{\frac{1}{N-r} - 1 + \eta}.$$

Since $2r > N$, $N - r > 1$ and $[k_{\text{tor}}(Y_0) : k_{\text{tor}}] \geq 1$, for η small enough we obtain

$$\deg B \ll_\eta ((h(C) + \deg C)[k_{\text{tor}}(C) : k_{\text{tor}}])^{\frac{r(N-r)}{2r-N} + \eta}. \tag{46}$$

So, from (45) we have

$$[k_{\text{tor}}(Y_0) : k_{\text{tor}}]\hat{h}(Y_0) \ll_\eta ((h(C) + \deg C)[k_{\text{tor}}(C) : k_{\text{tor}}])^{\frac{r}{2r-N} + \eta}; \tag{47}$$

while, using the right-hand side of (45) as a bound for $\hat{h}(Y_0)$ and (46) we get

$$\hat{h}(Y_0) \ll_\eta (h(C) + \deg C)^{\frac{r}{2r-N} + \eta}[k_{\text{tor}}(C) : k_{\text{tor}}]^{\frac{(N-r)}{2r-N} + \eta}$$

as required.

Having bounded $\deg B$ and $\hat{h}(Y_0)$, we now proceed to bound $[k(Y_0) : k]$ only in terms of C and E^N.

We use Siegel's lemma to construct an algebraic subgroup G of codimension $1 = \dim C$ defined over k, containing Y_0 and of controlled degree. The construction is exactly as in [Via03] Propositions 3 and 4. We present the steps of the proof.

We know that $\operatorname{End}(E)$ is an order in an imaginary quadratic field L with ring of integers $\mathcal{O}$. By minimality of $B + \zeta$, the coordinates of $Y_0 = (x_1, \ldots, x_N)$ generate an $\mathcal{O}$-module Γ of rank equal to the dimension of $B + \zeta$ which is $N - r$. Let $g_1, \ldots, g_{N-r}$ be generators of the free part of Γ which give the successive minima, and are chosen as in [Via03], Proposition 2. Then $\hat{h}(\sum_j \alpha_j g_j) \gg \sum_i |N_L(\alpha_j)|\hat{h}(g_j)$

for coefficients α_j in $\mathcal{O}$. In addition, like in the case of relative codimension one, the torsion part is generated by a torsion point T of exact order R.

Therefore we can write

$$x_i = \sum_j \alpha_{i,j} g_j + \beta_i T$$

with coefficients $\alpha_i, \beta_i \in \mathcal{O}$ and

$$N_L(\beta_i) \ll R^2.$$

As in Proposition 2 of [Via03], we have

$$\hat{h}(x_i) \gg \sum_j |N_L(\alpha_{i,j})| \, \hat{h}(g_j). \tag{48}$$

We define

$$\nu_j = (\alpha_{1,j}, \dots, \alpha_{N,j}) \quad \text{and} \quad |\nu_j| = \max_i |N_L(\alpha_{i,j})|.$$

Then

$$|\nu_j| \ll \frac{\hat{h}(Y_0)}{\hat{h}(g_j)}. \tag{49}$$

We want to find coefficients $a_i \in \mathcal{O}$ such that $\sum_i^N a_i x_i = 0$. This gives a linear system of $N - r + 1$ equations, obtained equating to zero the coefficients of g_j and of T. The system has coefficients in $\mathcal{O}$ and $N+1$ unknowns: the a_i's and one more unknown for the congruence relation arising from the torsion point.

We use the Siegel's lemma over $\mathcal{O}$ as stated in [BG06], Section 2.9. We get one equation with coefficients in $\mathcal{O}$; multiplying it by a constant depending only on E we may assume that it has coefficients in $\mathrm{End}(E)$. Thus it defines the sought-for algebraic subgroup G of degree

$$\deg G \ll \left((\max_i N_L(\beta_i)) \left(\prod_j^{N-r} |\nu_j| \right) \right)^{\frac{1}{r}}.$$

Let G_0 be a k-irreducible component of G passing through Y_0. Then

$$\deg G_0 \ll \left(R^2 \prod_j^{N-r} |\nu_j| \right)^{\frac{1}{r}}.$$

Since C is weak-transverse, the point Y_0 is a component of $C \cap G_0$. In addition C and G_0 are defined over k and Bézout's theorem gives

$$[k(Y_0) : k] \leq [k(C) : k] \deg C \deg G_0.$$

Hence

$$[k(Y_0) : k] \ll [k(C) : k] \deg C \left(R^2 \prod_j^{N-r} |\nu_j| \right)^{\frac{1}{r}}.$$

Using (49) we get

$$[k(Y_0) : k] \ll [k(C) : k] \deg C \left(R^2 \frac{\hat{h}(Y_0)^{N-r}}{\prod_j^{N-r} \hat{h}(g_j)} \right)^{\frac{1}{r}}. \tag{50}$$

Following step by step the proof of Proposition 4 in [Via03], using Theorem 4.5 in place of Theorem 1.3 of [DH00], we get

$$\prod_{i=1}^{N-r} \hat{h}(g_i) \gg_\eta \frac{1}{[k_{\rm tor}(Y_0):k_{\rm tor}]^{1+\eta}}. \tag{51}$$

By a result of Serre, recalled also in [Via03], Corollary 3, we know $[k(Y_0):k] \gg_\eta R^{2-\eta}$. Moreover from (47), the product $\hat{h}(Y_0)[k_{\rm tor}(Y_0):k_{\rm tor}]$ is bounded. Substituting these bounds in (50) and recalling that $r>2$, for η small enough we obtain

(52)

$$[k(Y_0):k] \ll_\eta ([k(C):k]\deg C)^{\frac{r}{r-1}+\eta}((h(C)+\deg C)[k_{\rm tor}(C):k_{\rm tor}])^{\frac{r(N-r)}{(2r-N)(r-1)}+\eta}.$$

Moreover, as in the proof of Theorem 6.1, we can choose ζ so that

$$[k(\zeta):k] \ll [k(Y_0):\mathbb{Q}],$$

and $\mathrm{ord}(\zeta) \ll_\eta [k(Y_0):\mathbb{Q}]^{\frac{N}{2}+\eta}$.

Substituting this and (46) in (42), we get the bound for $\deg H$.

Now, as in the last part of the proof of Theorem 6.1, for every $\eta>0$ a bound for the number S of non-torsion points Y_0 is given by

$$S \ll_\eta \deg C\ (\deg B)^{N+1+\eta}\mathrm{ord}(\zeta)^{2N+1}$$

and combining this with the previous results we obtain the bounds.

Finally we notice that we have bounded the degree $\deg H$ for H of fixed codimension r; letting r vary this proves that the subgroups H can be taken in a finite set $\{H_1,\dots,H_M\}$. A bound for M can be given effectively, as done in the proof of Theorem 6.2. □

In Theorem 9.1 we have stated a simplified version of the bounds for $\hat{h}(Y_0)$, $[k(Y_0):\mathbb{Q}]$ and S. This has been done for clarity, but a closer inspection of the proof shows that we have proved the following, sharper bounds.

Corollary 9.2. *With the same setting and notation of Theorem 9.1, take $Y_0 \in \mathcal{S}(C)$ is not a torsion point, and $Y_0 \in C\cap H$ with H of minimal dimension $N-r$. Then for any real $\eta>0$, there exist constants depending only on E^N and η such that*

$$\hat{h}(Y_0) \ll_\eta (h(C)+\deg C)^{\frac{r}{2r-N}+\eta}[k_{\rm tor}(C):k_{\rm tor}]^{\frac{(N-r)}{2r-N}+\eta},$$

$$[k(Y_0):\mathbb{Q}] \ll_\eta ([k(C):k]\deg C)^{\frac{r}{r-1}+\eta}((h(C)+\deg C)[k_{\rm tor}(C):k_{\rm tor}])^{\frac{r(N-r)}{(2r-N)(r-1)}+\eta}.$$

Moreover Y_0 belongs to a finite set of cardinality S bounded as

$$S \ll_\eta [k(C):k]^{B_1}(\deg C)^{B_1+1+\eta}((h(C)+\deg C)[k_{\rm tor}(C):k_{\rm tor}])^{B_2+\eta},$$

where

$$B_1 = \frac{rN(2N+1)}{2(r-1)},$$

$$B_2 = \frac{r(N-r)(2rN+2r-2+2N^2-N)}{2(2r-N)(r-1)}.$$

Appendix A. Sur une question d'orthogonalité dans les puissances de courbes elliptiques

We are indebted to Patrice Philippon for writing this appendix which is essentially the proof of Lemma 7.2. Let us mention that deeper properties of orthogonality in Mordell-Weil groups have been studied, see [Ber86].

Soit E une courbe elliptique définie sur le corps des nombres algébriques $\overline{\mathbb{Q}}$, considéré plongé dans le corps des nombres complexes $\mathbb{C}$. Soit $L = \mathrm{Frac}(\mathrm{End}(E))$ le corps des multiplications de E, qu'on considérera comme un sous-corps de $\overline{\mathbb{Q}} \subset \mathbb{C}$. On a $L = \mathbb{Q}$ ou $L = \mathbb{Q}(\sqrt{-D})$ pour un entier D positif, sans facteur carré.

On suppose E plongée dans un espace projectif par un diviseur ample et symétrique défini sur $\overline{\mathbb{Q}}$ et on note $\hat{h} : E\left(\overline{\mathbb{Q}}\right) \to \mathbb{R}$ la hauteur normalisée (*i.e.* de Néron-Tate) correspondante, qui satisfait $\hat{h}(\tau p) = N_{K/\mathbb{Q}}(\tau)\hat{h}(p)$ pour $\tau \in \mathrm{End}(E)$, $p \in E(\overline{\mathbb{Q}})$, et qui ne s'annule qu'aux points de torsion de E. L'accouplement de Néron-Tate associé s'écrit

$$\langle p, q\rangle_{NT} = \frac{1}{2}\left(\hat{h}(p+q) - \hat{h}(p) - \hat{h}(q)\right) \in \mathbb{R}$$

pour $p, q \in E(\overline{\mathbb{Q}})$, il induit naturellement une forme $\mathbb{Q}$-bilinéaire symétrique sur le L-espace vectoriel $E(\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. On introduit alors le produit scalaire (à valeurs dans $L \otimes_{\mathbb{Q}} \mathbb{R}$)

$$\langle p, q\rangle = \begin{cases} \langle p, q\rangle_{NT} & \text{si } L = \mathbb{Q} \\ \langle p, q\rangle_{NT} - \frac{1}{\sqrt{-D}}\langle p, \sqrt{-D}q\rangle_{NT} & \text{si } L = \mathbb{Q}(\sqrt{-D}), \end{cases}$$

dont on vérifie qu'il est L-sesqui-linéaire (relativement à la conjugaison sur L) et hermitien.

On étend ces produits scalaires à $E(\overline{\mathbb{Q}})^n \otimes_{\mathbb{Z}} \mathbb{Q}$ par les formules

$$\langle p, q\rangle_{NT} = \sum_{i=1}^{n} \langle p_i, q_i\rangle_{NT}$$
$$\langle p, q\rangle = \sum_{i=1}^{n} \langle p_i, q_i\rangle$$

pour $p = (p_1, \ldots, p_n)$ et $q = (q_1, \ldots, q_n)$ dans $E(\overline{\mathbb{Q}})^n$. Et on munit $\mathbb{C}^n$, identifié à l'espace tangent en l'origine de $E(\overline{\mathbb{C}})^n$, de son produit hermitien standard.

Lemma A.1. *Soient H et H' deux sous-groupes algébriques, connexes, de E^n, alors leurs espaces tangents à l'origine $\mathrm{T}H(\mathbb{C})$ et $\mathrm{T}H'(\mathbb{C})$ sont orthogonaux dans $\mathbb{C}^n$ si et seulement si $H(\overline{\mathbb{Q}})$ et $H'(\overline{\mathbb{Q}})$ sont orthogonaux dans E^n (pour le produit scalaire $\langle\cdot,\cdot\rangle$ ou de façon équivalente pour l'accouplement de Néron-Tate).*

Démonstration. Soient d et d' les dimensions respectives de H et H', il existe des homomorphismes $\varphi : E^d \to E^n$ et $\varphi' : E^{d'} \to E^n$ dont les images sont H et H' respectivement. On peut décrire ces homomorphismes sous la forme

$$(p_1,\dots,p_d)\mapsto p=\left(\sum_{j=1}^{d}a_{1,j}p_j,\dots,\sum_{j=1}^{d}a_{n,j}p_j\right)$$
$$(q_1,\dots,q_{d'})\mapsto q=\left(\sum_{k=1}^{d'}b_{1,k}q_k,\dots,\sum_{k=1}^{d'}b_{n,k}q_k\right)$$

et leurs applications tangentes $\mathrm{T}_\varphi:\mathbb{C}^d\to \mathrm{T}H(\mathbb{C})\subseteq\mathbb{C}^n$ et $\mathrm{T}'_\varphi:\mathbb{C}^{d'}\to \mathrm{T}H'(\mathbb{C})\subseteq\mathbb{C}^n$ par

$$(u_1,\dots,u_d)\mapsto u=\left(\sum_{j=1}^{d}a_{1,j}u_j,\dots,\sum_{j=1}^{d}a_{n,j}u_j\right)$$
$$(v_1,\dots,v_{d'})\mapsto v=\left(\sum_{k=1}^{d'}b_{1,k}v_k,\dots,\sum_{k=1}^{d'}b_{n,k}v_k\right)$$

où $a_{i,j},b_{i,k}\in\mathrm{End}(E)$ pour tous i,j,k. On calcule alors les produits scalaires en développant par sesquilinéarité

$$\langle u,v\rangle=\sum_{i=1}^{n}\sum_{j=1}^{d}\sum_{k=1}^{d'}a_{i,j}u_j\overline{b_{i,k}v_k}=\sum_{j=1}^{d}\sum_{k=1}^{d'}\left(\sum_{i=1}^{n}a_{i,j}\overline{b_{i,k}}\right)u_j\overline{v_k}$$

et

$$\langle p,q\rangle=\sum_{i=1}^{n}\sum_{j=1}^{d}\sum_{k=1}^{d'}a_{i,j}\overline{b_{i,k}}\langle p_j,q_k\rangle=\sum_{j=1}^{d}\sum_{k=1}^{d'}\left(\sum_{i=1}^{n}a_{i,j}\overline{b_{i,k}}\right)\langle p_j,q_k\rangle.$$

Si les espaces tangents en l'origine à H et H' sont orthogonaux (*resp.* si H et H' sont orthogonaux) on a $\langle u,v\rangle=0$ pour tous $u_1,\dots,u_d,v_1,\dots,v_{d'}\in\mathbb{C}$ (*resp.* $\langle p,q\rangle=\langle p,q\rangle_{\mathrm{NT}}=0$ pour tous $p_1,\dots,p_d,q_1,\dots,q_{d'}\in E(\overline{\mathbb{Q}})$. Appliqué avec $u_{j'}=v_{k'}=0$ pour $j'\neq j$, $k'\neq k$ et $u_j=v_k=u\neq 0$ (*resp.* $p_{j'}=q_{k'}=0$ pour $j'\neq j$, $k'\neq k$ et $p_j=q_k=p$ non de torsion), ceci entraîne les égalités $\sum_{i=1}^n a_{i,j}\overline{b_{i,k}}=0$, pour tous j,k. Réciproquement, ces dernières entraînent $\langle u,v\rangle=0$ et $\langle p,q\rangle=\langle p,q\rangle_{\mathrm{NT}}=0$ pour tous $u\in\mathrm{T}H(\mathbb{C})$, $v\in\mathrm{T}H'(\mathbb{C})$, $p\in H(\overline{\mathbb{Q}})$ et $q\in H'(\overline{\mathbb{Q}})$, ce qui établit l'équivalence énoncée. □

Acknowledgements

We wish to thank P. Philippon for useful discussions and for his kind communication of the appendix. We kindly thank A. Galateau for his communication of the constants in [Gal10]. We thank D. Bertrand, P. Habegger, M. Hindry and D. Masser for providing us useful references.

We warmly thank the FNS for the great support we receive in doing research in Mathematics. E. Viada also likes to thank the AIM and the organizers J. Pila and U. Zannier for supporting her participation to a conference on unlikely intersections in Pisa, where this work was inspired. S. Checcoli and F. Veneziano would like to address a special thank to their advisor U. Zannier for introducing them to mathematical research.

References

[AD03] Francesco Amoroso and Sinnou David, *Minoration de la hauteur normalisée dans un tore*, J. Inst. Math. Jussieu **2** (2003), no. 3, 335–381. MR 1990219 (2004m:11101)

[AD07] Francesco Amoroso and Emmanuel Delsinne, *Une minoration relative explicite pour la hauteur dans une extension d'une extension abélienne*, Diophantine geometry, CRM Series, vol. 4, Ed. Norm., Pisa, 2007, pp. 1–24. MR 2349643 (2008i:11091)

[AV12] Francesco Amoroso and Evelina Viada, *Small points on rational subvarieties of tori*, Comment. Math. Helv. **87** (2012), no. 2, 355–383. MR 2914852

[AZ00] Francesco Amoroso and Umberto Zannier, *A relative Dobrowolski lower bound over abelian extensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), no. 3, 711–727. MR 1817715 (2003a:11078)

[Ber86] Daniel Bertrand, *Relations d'orthogonalité sur les groupes de Mordell-Weil*, Séminaire de théorie des nombres, Paris 1984–85, Progr. Math., vol. 63, Birkhäuser Boston, Boston, MA, 1986, pp. 33–39. MR 897339 (88h:11037)

[Ber87] ———, *Minimal heights and polarizations on abelian varieties*, MSRI, preprint 06220-87, June 1987.

[BG06] Enrico Bombieri and Walter Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006. MR 2216774 (2007a:11092)

[BGEP09] Attila Bérczes, Kálmán Győry, Jan-Hendrik Evertse, and Corentin Pontreau, *Effective results for points on certain subvarieties of tori*, Math. Proc. Cambridge Philos. Soc. **147** (2009), no. 1, 69–94. MR 2507310 (2010k:14031)

[BHMZ10] Enrico Bombieri, Philipp Habegger, David Masser, and Umberto Zannier, *A note on Maurin's theorem*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **21** (2010), no. 3, 251–260. MR 2677603 (2011f:11078)

[BMZ99] Enrico Bombieri, David Masser, and Umberto Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Internat. Math. Res. Notices (1999), no. 20, 1119–1140. MR 1728021 (2001c:11081)

[BMZ07] ———, *Anomalous subvarieties—structure theorems and applications*, Int. Math. Res. Not. IMRN (2007), no. 19, Art. ID rnm057, 33. MR 2359537 (2008k:11060)

[BZ96] Enrico Bombieri and Umberto Zannier, *Heights of algebraic points on subvarieties of abelian varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23** (1996), no. 4, 779–792 (1997). MR 1469574 (98j:11043)

[Car09] María Carrizosa, *Petits points et multiplication complexe*, Int. Math. Res. Not. IMRN (2009), no. 16, 3016–3097. MR 2533796 (2011c:11102)

[CVV12] Sara Checcoli, Francesco Veneziano, and Evelina Viada, *A sharp Bogomolov-type bound*, NYJM **18** (2012), 891–910.

[Del09] Emmanuel Delsinne, *Le problème de Lehmer relatif en dimension supérieure*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 6, 981–1028. MR 2567747 (2011a:11130)

[DH00] Sinnou David and Marc Hindry, *Minoration de la hauteur de Néron-Tate sur les variétés abéliennes de type C. M*, J. Reine Angew. Math. **529** (2000), 1–74. MR 1799933 (2001j:11054)

[DP07] Sinnou David and Patrice Philippon, *Minorations des hauteurs normalisées des sous-variétés des puissances des courbes elliptiques*, Int. Math. Res. Pap. IMRP (2007), no. 3, Art. ID rpm006, 113. MR 2355454 (2008h:11068)

[Gal10] Aurélien Galateau, *Une minoration du minimum essentiel sur les variétés abéliennes*, Comment. Math. Helv. **85** (2010), no. 4, 775–812. MR 2718139 (2011i:11110)

[Hab08] Philipp Habegger, *Intersecting subvarieties of* $\mathbf{G}_m^n$ *with algebraic subgroups*, Math. Ann. **342** (2008), no. 2, 449–466. MR 2425150 (2009f:14044)

[Hab09] ———, *Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension*, Invent. Math. **176** (2009), no. 2, 405–447. MR 2495768 (2010f:11109)

[Hru01] Ehud Hrushovski, *The Manin-Mumford conjecture and the model theory of difference fields*, Ann. Pure Appl. Logic **112** (2001), no. 1, 43–115. MR 1854232 (2003d:03061)

[Mau08] Guillaume Maurin, *Courbes algébriques et équations multiplicatives*, Math. Ann. **341** (2008), no. 4, 789–824. MR 2407327 (2009g:14026)

[McC94] William McCallum, *On the method of Coleman and Chabauty*, Math. Ann. **299** (1994), no. 3, 565–596. MR 1282232 (95c:11079)

[MW93] David Masser and Gisbert Wüstholz, *Periods and minimal abelian subvarieties*, Ann. of Math. (2) **137** (1993), no. 2, 407–458. MR 1207211 (94g:11040)

[Phi91] Patrice Philippon, *Sur des hauteurs alternatives. I*, Math. Ann. **289** (1991), no. 2, 255–283. MR 1092175 (92m:11061)

[Phi95] ———, *Sur des hauteurs alternatives. III*, J. Math. Pures Appl. (9) **74** (1995), no. 4, 345–365. MR 1341770 (97a:11098)

[PM10] Bjorn Poonen and William McCallum, *The method of Chabauty and Coleman*, to appear in *Panoramas et Synthèses*, Société Math. de France. Preprint, `http://www-math.mit.edu/~poonen/papers/chabauty.pdf`, June 2010.

[Rat04] Nicolas Ratazzi, *Théorème de Dobrowolski-Laurent pour les extensions abéliennes sur une courbe elliptique à multiplication complexe*, Int. Math. Res. Not. (2004), no. 58, 3121–3152. MR 2098701 (2005k:11116)

[Rém00] Gaël Rémond, *Décompte dans une conjecture de Lang*, Invent. Math. **142** (2000), no. 3, 513–545. MR 1804159 (2002f:14058)

[Ser72] Jean-Pierre Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. MR 0387283 (52 #8126)

[Ser89] ———, *Lectures on the Mordell-Weil theorem*, Aspects of Mathematics, E15, Friedr. Vieweg & Sohn, Braunschweig, 1989, Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt. MR 1002324 (90e:11086)

[Via03] Evelina Viada, *The intersection of a curve with algebraic subgroups in a product of elliptic curves*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), no. 1, 47–75. MR 1990974 (2004c:11099)

[Via08] ———, *The intersection of a curve with a union of translated codimension-two subgroups in a power of an elliptic curve*, Algebra Number Theory **2** (2008), no. 3, 249–298. MR 2407116 (2009f:11079)

[Via10] ———, *Lower bounds for the normalized height and non-dense subsets of subvarieties of abelian varieties*, Int. J. Number Theory **6** (2010), no. 3, 471–499. MR 2652892 (2011i:11111)

[Zha98] Shou-Wu Zhang, *Equidistribution of small points on abelian varieties*, Ann. of Math. (2) **147** (1998), no. 1, 159–165. MR 1609518 (99e:14032)

Mathematisches Institut, Universität Basel, Rheinsprung 21, CH-4051 Basel, Switzerland.

E-mail address: `sara.checcoli@unibas.ch`

Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3-5, D-37073 Göttingen, Germany.

E-mail address: `francesco.veneziano@unibas.ch`

Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3-5, D-37073 Göttingen, Germany.

E-mail address: `evelina.viada@unibas.ch`